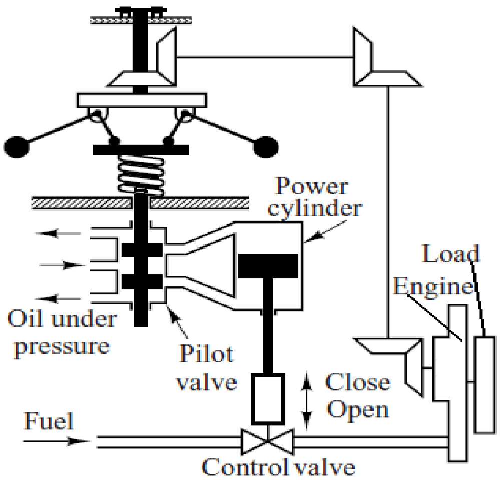
Cairo University
Faculty of Engineering
Mech. Power Department

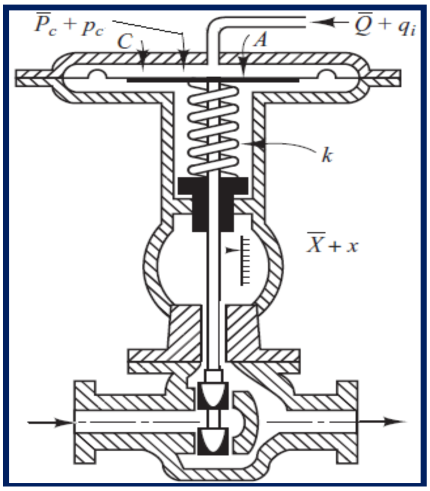
ACC Virtual Labs
Automatic Control Circuits & Virtual Labs for Mechanical Power Systems
معمل التحكم الأتوماتيكى و المعامل الإفتراضية لأنظمة القوى الميكانيكية

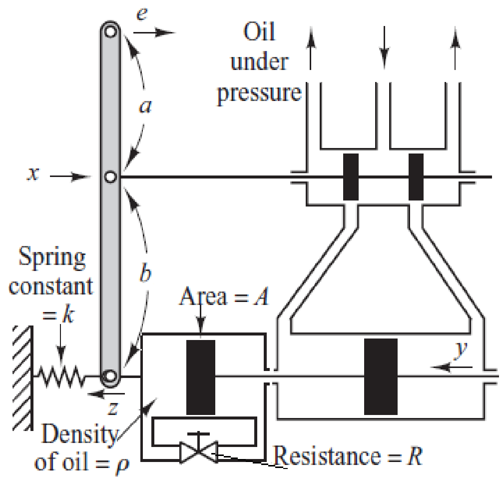
تطبيقات نظرية التحكم الأوتوماتيكى فى نظم القوى الميكانيكية

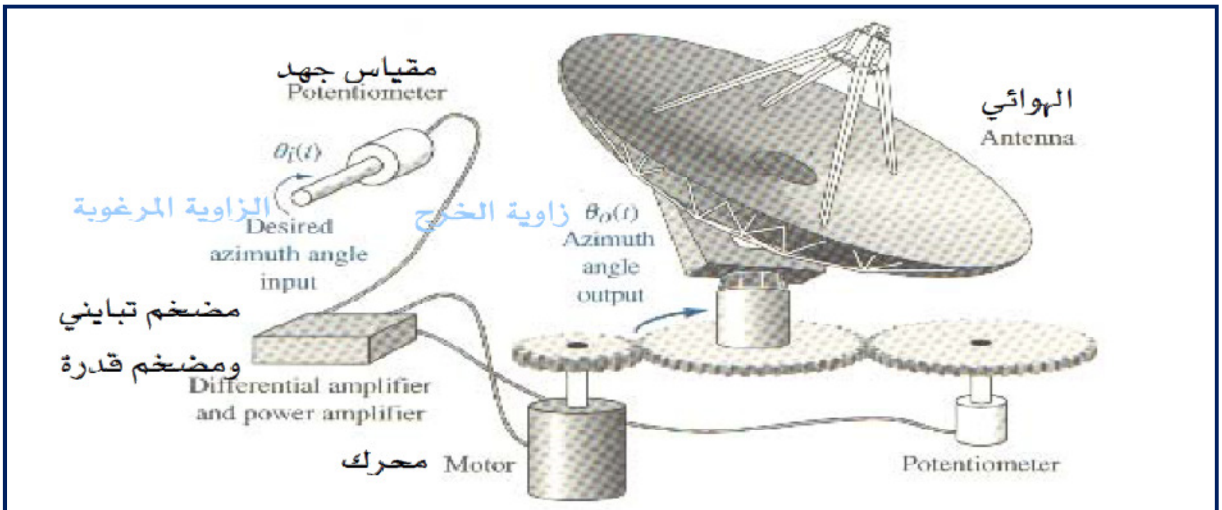
Applications of Control Theory in Mech Power Systems

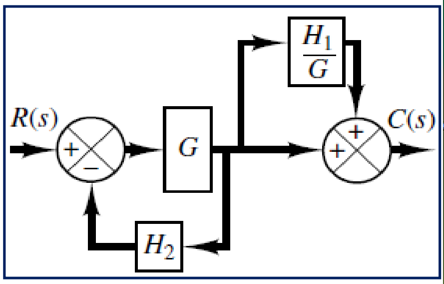
Dr. Mohsen Soliman

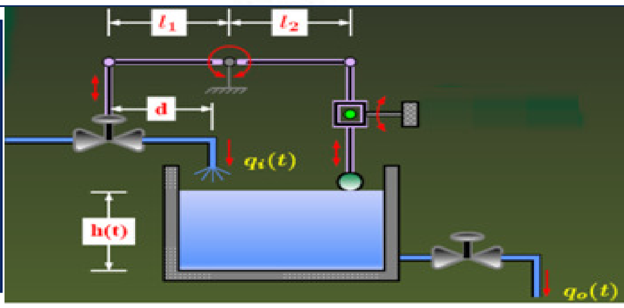


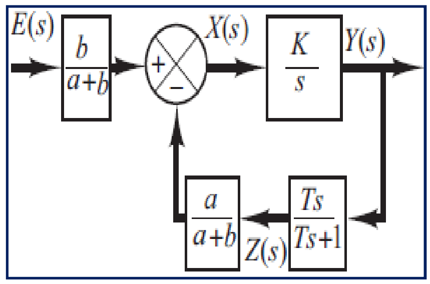












********** Part-2 (English-chapters 2+3+4 of the Book) **********

ملاحظة: الأجزاء السابقة تضمنت ما يلى:

محتوى وأهداف الجزء الأول:مقدمة عامة،وصف وتعريف أنواع نظم التحكم، تحديد المكونات الميكانيكية والهوائية والهيدروليكية والكهربائية،طرق النمذجة الرياضية وتحديد دالة التحويل، وصف النظام ذوالحلقة المغلقة، تحديدالاستجابة الوقتية لمنظومات التحكم (جزء مختصرخاص بالدبلوم لمقرر مكق561).

محتوى وأهداف الجزء الثانى(هذا الجزء ويشمل الفصل الثانى والثالث والرابع فقط): مراجعة عامة لكل المفاهيم الأساسية لنظرية التحكم - أمثلة تطبيقية على نظم ميكانيكية (لاحظ وجود أمثلة إضافية فى الجزءالأولControlTheoryBookعربى-part1.pdf) وتطبيقات كهربية على PID-controller تتضمن تحويلات لابلاس وإستنتاج دالة التحويل وتمييز درجة المعادلات التفاضلية والرياضية التى تمثل العديد من منظومات التحكم الأوتوماتيكى وخاصة التى تدخل فى مجال نظم القوى الميكانيكية.

محتوى وأهداف الجزء الثالث (يشمل الفصل الخامس التالى فقط): مراجعة رد الفعل العابر وخصائصه الهامة طبقاً لدرجة النماذج الرياضية لمنظومات التحكم – تحليل الخطأ فى أنظمة التحكم وتأثيرات نوعية المنظمات على أداء المنظومة (ويتضمن ذلك تحديد إشارة الخطأ التى تنشأ فى دائرة التحكم المغلقة حيث نقوم بتحليلها ومن ثم نوضح طرق إستخدام أنواع المنظمات المختلفة للحصول على أداء مثالى للنظام) – تحليل إستقرار أنظمة التحكم الأوتوماتيكى ودراسة إستقرار العمليات فى منظومات التحكم الآلى (ويتضمن ذلك دراسة إستقرار نظم التحكم الأوتوماتيكية بالطريقة الجبرية Routh method حتى نتمكن من الحصول على منظومة تحكم مستقرة فى أدائها).

Contents

2 Mathematical Modeling of Control Systems

2–1 INTRODUCTION

In studying control systems the reader must be able to model dynamic systems in mathematical terms and analyze their dynamic characteristics. A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on one's perspective.

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system—for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving reasonable mathematical models is the most important part of the entire analysis of control systems.

Throughout this book we assume that the principle of causality applies to the systems considered. This means that the current output of the system (the output at time $t = 0$) depends on the past input (the input for $t < 0$) but does not depend on the future input (the input for $t > 0$).

Mathematical Models. Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations. On the other hand, for the transient-response or frequency-response analysis of single-input, single-output, linear, time-invariant systems, the transfer-function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

Simplicity Versus Accuracy. In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis. In deriving a reasonably simplified mathematical model, we frequently find it necessary to ignore certain inherent physical properties of the system. In particular, if a linear lumped-parameter mathematical model (that is, one employing ordinary differential equations) is desired, it is always necessary to ignore certain nonlinearities and distributed parameters that may be present in the physical system. If the effects that these ignored properties have on the response are small, good agreement will be obtained between the results of the analysis of a mathematical model and the results of the experimental study of the physical system.

In general, in solving a new problem, it is desirable to build a simplified model so that we can get a general feeling for the solution. A more complete mathematical model may then be built and used for a more accurate analysis.

We must be well aware that a linear lumped-parameter model, which may be valid in low-frequency operations, may not be valid at sufficiently high frequencies, since the neglected property of distributed parameters may become an important factor in the dynamic behavior of the system. For example, the mass of a spring may be neglected in low-frequency operations, but it becomes an important property of the system at high frequencies. (For the case where a mathematical model involves considerable errors, robust control theory may be applied. Robust control theory is presented in Chapter 10.)

Linear Systems. A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. It is this principle that allows one to build up complicated solutions to the linear differential equation from simple solutions.

In an experimental investigation of a dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered linear.

Linear Time-Invariant Systems and Linear Time-Varying Systems. A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant differential equations—that is, constant-coefficient differential equations. Such systems are called *linear time-invariant* (or *linear constant-coefficient*) systems. Systems that are represented by differential equations whose coefficients are functions of time are called *linear time-varying* systems. An example of a time-varying control system is a spacecraft control system. (The mass of a spacecraft changes due to fuel consumption.)

Outline of the Chapter. Section 2–1 has presented an introduction to the mathematical modeling of dynamic systems. Section 2–2 presents the transfer function and impulse-response function. Section 2–3 introduces automatic control systems and Section 2–4 discusses concepts of modeling in state space. Section 2–5 presents state-space representation of dynamic systems. Section 2–6 discusses transformation of mathematical models with MATLAB. Finally, Section 2–7 discusses linearization of nonlinear mathematical models.

2–2 TRANSFER FUNCTION AND IMPULSE-RESPONSE FUNCTION

In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equations. We begin by defining the transfer function and follow with a derivation of the transfer function of a differential equation system. Then we discuss the impulse-response function.

Transfer Function. The *transfer function* of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an *nth-order system*.

Comments on Transfer Function. The applicability of the concept of the transfer function is limited to linear, time-invariant, differential equation systems. The transfer function approach, however, is extensively used in the analysis and design of such systems. In what follows, we shall list important comments concerning the transfer function. (Note that a system referred to in the list is one described by a linear, time-invariant, differential equation.)

1. The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
2. The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)
4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.
5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

Convolution Integral For linear, time-invariant system the transfer function $G(s)$ is

$$G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ is the Laplace transform of the input to the system and $Y(s)$ is the Laplace transform of the output of the system, where we assume that all initial conditions involved are zero. It follows that the output $Y(s)$ can be written as the product of $G(s)$ and $X(s)$, or

$$Y(s) = G(s)X(s) \qquad (2–1)$$

Note that multiplication in the complex domain is equivalent to convolution in the time domain (see Appendix A), so the inverse Laplace transform of Equation (2–1) is given by the following convolution integral:

$$y(t) = \int_0^t x(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)x(t-\tau)\,d\tau$$

where both $g(t)$ and $x(t)$ are 0 for $t < 0$.

Impulse-Response Function. Consider the output (response) of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is

$$Y(s) = G(s) \qquad (2–2)$$

The inverse Laplace transform of the output given by Equation (2–2) gives the impulse response of the system. The inverse Laplace transform of $G(s)$, or

$$\mathscr{L}^{-1}[G(s)] = g(t)$$

is called the impulse-response function. This function $g(t)$ is also called the weighting function of the system.

The impulse-response function $g(t)$ is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function. Therefore, the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response. (In practice, a pulse input with a very short duration compared with the significant time constants of the system can be considered an impulse.)

2–3 AUTOMATIC CONTROL SYSTEMS

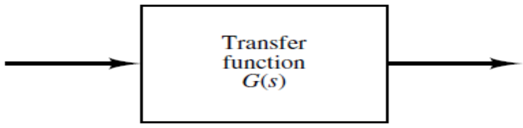
A control system may consist of a number of components. To show the functions performed by each component, in control engineering, we commonly use a diagram called the *block diagram*. This section first explains what a block diagram is. Next, it discusses introductory aspects of automatic control systems, including various control actions. Then, it presents a method for obtaining block diagrams for physical systems, and, finally, discusses techniques to simplify such diagrams.

Block Diagrams. A *block diagram* of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. Differing from a purely abstract mathematical representation, a block diagram has the advantage of indicating more realistically the signal flows of the actual system.

In a block diagram all system variables are linked to each other through functional blocks. The *functional* block or simply *block* is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that the signal can pass only in the direction of the arrows. Thus a block diagram of a control system explicitly shows a unilateral property.

Figure 2–1 shows an element of the block diagram. The arrowhead pointing toward the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as *signals*.

Figure 2–1 Element of a block diagram.



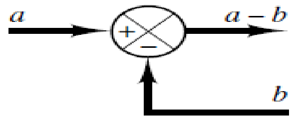


Figure 2–2 Summing point.

Note that the dimension of the output signal from the block is the dimension of the input signal multiplied by the dimension of the transfer function in the block.

The advantages of the block diagram representation of a system are that it is easy to form the overall block diagram for the entire system by merely connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

In general, the functional operation of the system can be visualized more readily by examining the block diagram than by examining the physical system itself. A block diagram contains information concerning dynamic behavior, but it does not include any information on the physical construction of the system. Consequently, many dissimilar and unrelated systems can be represented by the same block diagram.

It should be noted that in a block diagram the main source of energy is not explicitly shown and that the block diagram of a given system is not unique. A number of different block diagrams can be drawn for a system, depending on the point of view of the analysis.

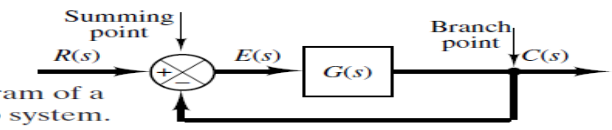
Summing Point. Referring to Figure 2–2, a circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

Branch Point. A *branch point* is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block Diagram of a Closed-Loop System. Figure 2–3 shows an example of a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point, where it is compared with the reference input $R(s)$. The closed-loop nature of the system is clearly indicated by the figure. The output of the block, $C(s)$ in this case, is obtained by multiplying the transfer function $G(s)$ by the input to the block, $E(s)$. Any linear control system may be represented by a block diagram consisting of blocks, summing points, and branch points.

When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. For example, in a temperature control system, the output signal is usually the controlled temperature. The output signal, which has the dimension of temperature, must be converted to a force or position or voltage before it can be compared with the input signal. This conversion is accomplished by the feedback element whose transfer function is $H(s)$, as shown in Figure 2–4. The role of the feedback element is to modify the output before it is compared with the input. (In most cases the feedback element is a sensor that measures

Figure 2–3 Block diagram of a closed-loop system.



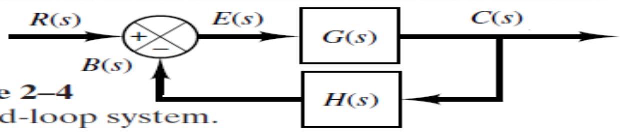


Figure 2–4 Closed-loop system.

the output of the plant. The output of the sensor is compared with the system input, and the actuating error signal is generated.) In the present example, the feedback signal that is fed back to the summing point for comparison with the input is $B(s) = H(s)C(s)$.

Open-Loop Transfer Function and Feedforward Transfer Function. Referring to Figure 2–4, the ratio of the feedback signal $B(s)$ to the actuating error signal $E(s)$ is called the *open-loop transfer function*. That is,

$$\text{Open-loop transfer function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

The ratio of the output $C(s)$ to the actuating error signal $E(s)$ is called the *feedforward transfer function*, so that

$$\text{Feedforward transfer function} = \frac{C(s)}{E(s)} = G(s)$$

If the feedback transfer function $H(s)$ is unity, then the open-loop transfer function and the feedforward transfer function are the same.

Closed-Loop Transfer Function. For the system shown in Figure 2–4, the output $C(s)$ and input $R(s)$ are related as follows: since $C(s) = G(s)E(s)$

$$E(s) = R(s) - B(s) = R(s) - H(s)C(s)$$

eliminating $E(s)$ from these equations gives $C(s) = G(s)[R(s) - H(s)C(s)]$

or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad (2\text{–}3)$$

The transfer function relating $C(s)$ to $R(s)$ is called the *closed-loop transfer function*. It relates the closed-loop system dynamics to the dynamics of the feedforward elements and feedback elements. From Equation (2–3), $C(s)$ is given by

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Thus the output of the closed-loop system clearly depends on both the closed-loop transfer function and the nature of the input.

Obtaining Cascaded, Parallel, and Feedback (Closed-Loop) Transfer Functions with MATLAB. In control-systems analysis, we frequently need to calculate the cascaded transfer functions, parallel-connected transfer functions, and feedback-connected (closed-loop) transfer functions. MATLAB has convenient commands to obtain the cascaded, parallel, and feedback (closed-loop) transfer functions.

Suppose that there are two components $G_1(s)$ and $G_2(s)$ connected differently as shown in Figure 2–5 (a), (b), and (c), where

$$G_1(s) = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{\text{num2}}{\text{den2}}$$

To obtain the transfer functions of the cascaded system, parallel system, or feedback (closed-loop) system, the following commands may be used:

```
[num, den] = series(num1,den1,num2,den2)
[num, den] = parallel(num1,den1,num2,den2)
[num, den] = feedback(num1,den1,num2,den2)
```

As an example, consider the case where

$$G_1(s) = \frac{10}{s^2 + 2s + 10} = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{5}{s + 5} = \frac{\text{num2}}{\text{den2}}$$

MATLAB Program 2–1 gives $C(s)/R(s) = \text{num/den}$ for each arrangement of $G_1(s)$ and $G_2(s)$. Note that the command

```
printsys(num,den)
```

displays the num/den [that is, the transfer function $C(s)/R(s)$] of the system considered.

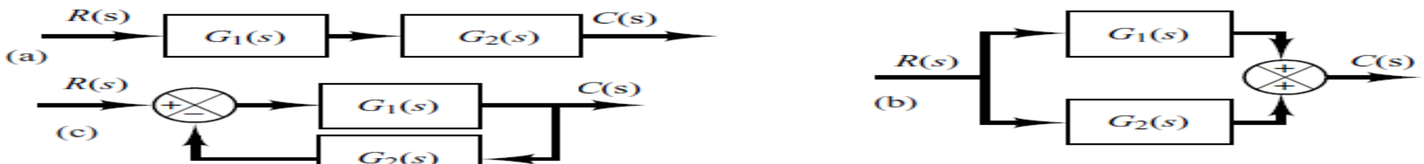


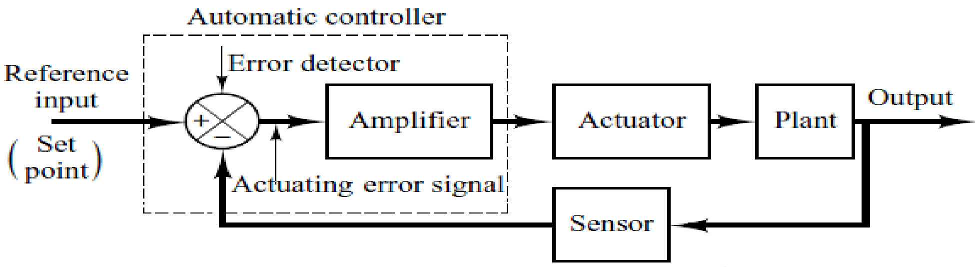
Figure 2–5
(a) Cascaded system; (b) parallel system; (c) feedback (closed-loop) system

MATLAB Program 2–1

```
num1 = [10];
den1 = [1  2  10];
num2 = [5];
den2 = [1  5];
[num, den] = series(num1,den1,num2,den2);
printsys(num,den)
num/den =          50
          ----------------------
          s^3 + 7s^2 + 20s + 50
[num, den] = parallel(num1,den1,num2,den2);
printsys(num,den)
num/den =     5s^2 + 20s + 100
          ----------------------
          s^3 + 7s^2 + 20s + 50
[num, den] = feedback(num1,den1,num2,den2);
printsys(num,den)
num/den =         10s + 50
          -----------------------
          s^3 + 7s^2 + 20s + 100
```

Automatic Controllers. An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value. The manner in which the automatic controller produces the control signal is called the *control action*. Figure 2–6 is a block diagram of an industrial control system, which

Figure 2–6
Block diagram of an industrial control system, which consists of an automatic controller, an actuator, a plant, and a sensor (measuring element).



consists of an automatic controller, an actuator, a plant, and a sensor (measuring element). The controller detects the actuating error signal, which is usually at a very low power level, and amplifies it to a sufficiently high level. The output of an automatic controller is fed to an actuator, such as an electric motor, a hydraulic motor, or a pneumatic motor or valve. (The actuator is a power device that produces the input to the plant according to the control signal so that the output signal will approach the reference input signal.)

The sensor or measuring element is a device that converts the output variable into another suitable variable, such as a displacement, pressure, voltage, etc., that can be used to compare the output to the reference input signal. This element is in the feedback path of the closed-loop system. The set point of the controller must be converted to a reference input with the same units as the feedback signal from the sensor or measuring element.

Classifications of Industrial Controllers. Most industrial controllers may be classified according to their control actions as:

1. Two-position or on–off controllers
2. Proportional controllers
3. Integral controllers
4. Proportional-plus-integral controllers
5. Proportional-plus-derivative controllers
6. Proportional-plus-integral-plus-derivative controllers

Most industrial controllers use electricity or pressurized fluid such as oil or air as power sources. Consequently, controllers may also be classified according to the kind of power employed in the operation, such as pneumatic controllers, hydraulic controllers, or electronic controllers. What kind of controller to use must be decided based on the nature of the plant and the operating conditions, including such considerations as safety, cost, availability, reliability, accuracy, weight, and size.

Two-Position or On–Off Control Action. In a two-position control system, the actuating element has only two fixed positions, which are, in many cases, simply on and off. Two-position or on–off control is relatively simple and inexpensive and, for this reason, is very widely used in both industrial and domestic control systems.

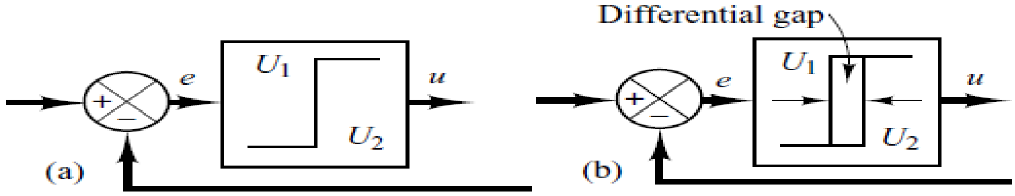
Let the output signal from the controller be $u(t)$ and the actuating error signal be $e(t)$. In two-position control, the signal $u(t)$ remains at either a maximum or minimum value, depending on whether the actuating error signal is positive or negative, so that

$$u(t) = U_1, \quad \text{for } e(t) > 0 \qquad u(t) = U_2, \quad \text{for } e(t) < 0$$

where U_1 and U_2 are constants. The minimum value U_2 is usually either zero or $-U_1$. Two-position controllers are generally electrical devices, and an electric solenoid-operated valve is widely used in such controllers. Pneumatic proportional controllers with very high gains act as two-position controllers and are sometimes called pneumatic two-position controllers.

Figures 2–7(a) and (b) show the block diagrams for two-position or on–off controllers. The range through which the actuating error signal must move before the switching occurs

Figure 2–7
(a) Block diagram of an on–off controller; (b) block diagram of an on–off controller with differential gap.



is called the *differential gap*. A differential gap is indicated in Figure 2–7(b). Such a differential gap causes the controller output $u(t)$ to maintain its present value until the actuating error signal has moved slightly beyond the zero value. In some cases, the differential gap is a result of unintentional friction and lost motion; however, quite often it is intentionally provided in order to prevent too-frequent operation of the on–off mechanism.

Consider the liquid-level control system shown in Figure 2–8(a), where the electromagnetic valve shown in Figure 2–8(b) is used for controlling the inflow rate. This valve is either open or closed. With this two-position control, the water inflow rate is either a positive constant or zero. As shown in Figure 2–9, the output signal continuously moves between the two limits required to cause the actuating element to move from one fixed position to the other. Notice that the output curve follows one of two exponential curves, one corresponding to the filling curve and the other to the emptying curve. Such output oscillation between two limits is a typical response characteristic of a system under two-position control.

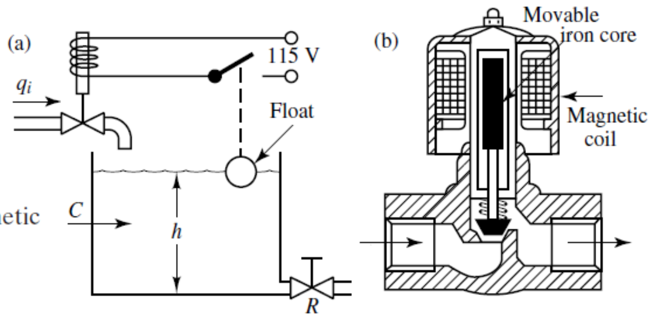


Figure 2–8
(a) Liquid-level control system; (b) electromagnetic valve.

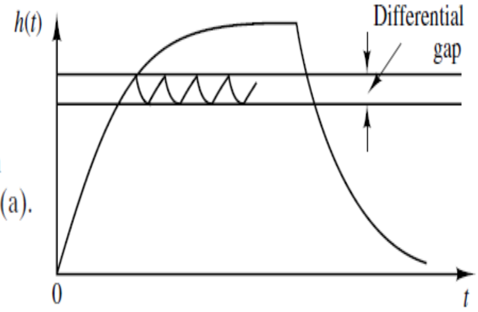


Figure 2–9
Level $h(t)$-versus-t curve for the system shown in Figure 2–8(a).

From Figure 2–9, we notice that the amplitude of the output oscillation can be reduced by decreasing the differential gap. The decrease in the differential gap, however, increases the number of on–off switchings per minute and reduces the useful life of the component. The magnitude of the differential gap must be determined from such considerations as the accuracy required and the life of the component.

Proportional Control Action. For a controller with proportional control action, the relationship between the output of the controller $u(t)$ and the actuating error signal $e(t)$ is

$$u(t) = K_p e(t)$$

or, in Laplace-transformed quantities,

$$\frac{U(s)}{E(s)} = K_p$$

where K_p is termed the proportional gain.

Whatever the actual mechanism may be and whatever the form of the operating power, the proportional controller is essentially an amplifier with an adjustable gain.

Integral Control Action. In a controller with integral control action, the value of the controller output $u(t)$ is changed at a rate proportional to the actuating error signal $e(t)$. That is, $\dfrac{du(t)}{dt} = K_i e(t)$ or $u(t) = K_i \int_0^t e(t)\,dt$ where K_i is an adjustable constant. The transfer function of the integral controller is $\dfrac{U(s)}{E(s)} = \dfrac{K_i}{s}$

Proportional-Plus-Integral Control Action. The control action of a proportional-plus-integral controller is defined by $u(t) = K_p e(t) + \dfrac{K_p}{T_i}\int_0^t e(t)\,dt$ or the transfer function of the controller is $\dfrac{U(s)}{E(s)} = K_p\left(1 + \dfrac{1}{T_i s}\right)$ where T_i is called the *integral time*.

Proportional-Plus-Derivative Control Action. The control action of a proportional-plus-derivative controller is defined by $u(t) = K_p e(t) + K_p T_d \dfrac{de(t)}{dt}$ and the transfer function is $\dfrac{U(s)}{E(s)} = K_p(1 + T_d s)$ where T_d is called the *derivative time*.

Proportional-Plus-Integral-Plus-Derivative Control Action. The combination of proportional control action, integral control action, and derivative control action is termed proportional-plus-integral-plus-derivative control action. It has the advantages of each of the three individual control actions. The equation of a controller with this combined action is given by $u(t) = K_p e(t) + \dfrac{K_p}{T_i}\int_0^t e(t)\,dt + K_p T_d \dfrac{de(t)}{dt}$ or the transfer function is $\dfrac{U(s)}{E(s)} = K_p\left(1 + \dfrac{1}{T_i s} + T_d s\right)$ where K_p is the proportional gain, T_i is the integral time, and T_d is the derivative time. The block diagram of a proportional-plus-integral-plus-derivative controller is shown in Figure 2–10.

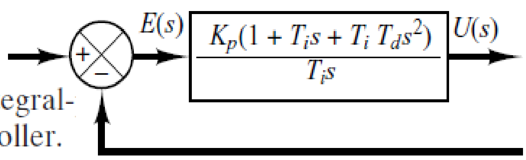


Figure 2–10
Block diagram of a proportional-plus-integral-plus-derivative controller.

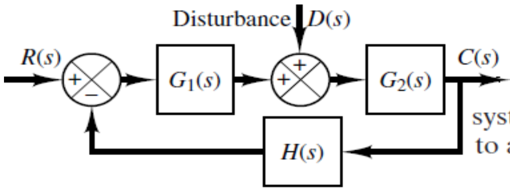


Figure 2–11
Closed-loop system subjected to a disturbance.

Closed-Loop System Subjected to a Disturbance. Figure 2–11 shows a closed-loop system subjected to a disturbance. When two inputs (the reference input and disturbance) are present in a linear time-invariant system, each input can be treated independently of the other; and the outputs corresponding to each input alone can be added to give the complete output. The way each input is introduced into the system is shown at the summing point by either a plus or minus sign.

Consider the system shown in Figure 2–11. In examining the effect of the disturbance $D(s)$, we may assume that the reference input is zero; we may then calculate the response $C_D(s)$ to the disturbance only. This response can be found from

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

On the other hand, in considering the response to the reference input $R(s)$, we may assume that the disturbance is zero. Then the response $C_R(s)$ to the reference input $R(s)$ can be obtained from

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

The response to the simultaneous application of the reference input and disturbance can be obtained by adding the two individual responses. In other words, the response $C(s)$ due to the simultaneous application of the reference input $R(s)$ and disturbance $D(s)$ is given by

$$C(s) = C_R(s) + C_D(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + D(s)]$$

Consider now the case where $|G_1(s)H(s)| \gg 1$ and $|G_1(s)G_2(s)H(s)| \gg 1$. In this case, the closed-loop transfer function $C_D(s)/D(s)$ becomes almost zero, and the effect of the disturbance is suppressed. This is an advantage of the closed-loop system.

On the other hand, the closed-loop transfer function $C_R(s)/R(s)$ approaches $1/H(s)$ as the gain of $G_1(s)G_2(s)H(s)$ increases. This means that if $|G_1(s)G_2(s)H(s)| \gg 1$, then the closed-loop transfer function $C_R(s)/R(s)$ becomes independent of $G_1(s)$ and $G_2(s)$ and inversely proportional to $H(s)$, so that the variations of $G_1(s)$ and $G_2(s)$ do not affect the closed-loop transfer function $C_R(s)/R(s)$. This is another advantage of the closed-loop system. It can easily be seen that any closed-loop system with unity feedback, $H(s) = 1$, tends to equalize the input and output.

Procedures for Drawing a Block Diagram. To draw a block diagram for a system, first write the equations that describe the dynamic behavior of each component. Then take the Laplace transforms of these equations, assuming zero initial conditions, and represent each Laplace-transformed equation individually in block form. Finally, assemble the elements into a complete block diagram.

As an example, consider the RC circuit shown in Figure 2–12(a). The equations for this circuit are

$$i = \frac{e_i - e_o}{R} \qquad (2\text{–}4) \qquad\qquad e_o = \frac{\int i\,dt}{C} \qquad (2\text{–}5)$$

The Laplace transforms of Equations (2–4) and (2–5), with zero initial condition, become

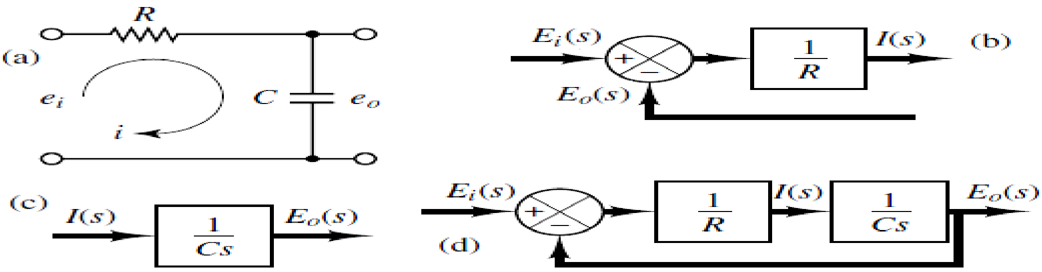
$$I(s) = \frac{E_i(s) - E_o(s)}{R} \qquad (2\text{–}6) \qquad\qquad E_o(s) = \frac{I(s)}{Cs} \qquad (2\text{–}7)$$

Equation (2–6) represents a summing operation, and the corresponding diagram is shown in Figure 2–12(b). Equation (2–7) represents the block as shown in Figure 2–12(c). Assembling these two elements, we obtain the overall block diagram for the system as shown in Figure 2–12(d).

Block Diagram Reduction. It is important to note that blocks can be connected in series only if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block.

Any number of cascaded blocks representing nonloading components can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions.

Figure 2–12
(a) *RC* circuit;
(b) block diagram representing Equation (2–6);
(c) block diagram representing Equation (2–7);
(d) block diagram of the *RC* circuit.



A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangement. Simplification of the block diagram by rearrangements considerably reduces the labor needed for subsequent mathematical analysis. It should be noted, however, that as the block diagram is simplified, the transfer functions in new blocks become more complex because new poles and new zeros are generated.

Consider the system shown in Figure 2–13(a). Simplify this diagram.

By moving the summing point of the negative feedback loop containing H_2 outside the positive feedback loop containing H_1, we obtain Figure 2–13(b). Eliminating the positive feedback loop, we have Figure 2–13(c). The elimination of the loop containing H_2/G_1 gives Figure 2–13(d). Finally, eliminating the feedback loop results in Figure 2–13(e).

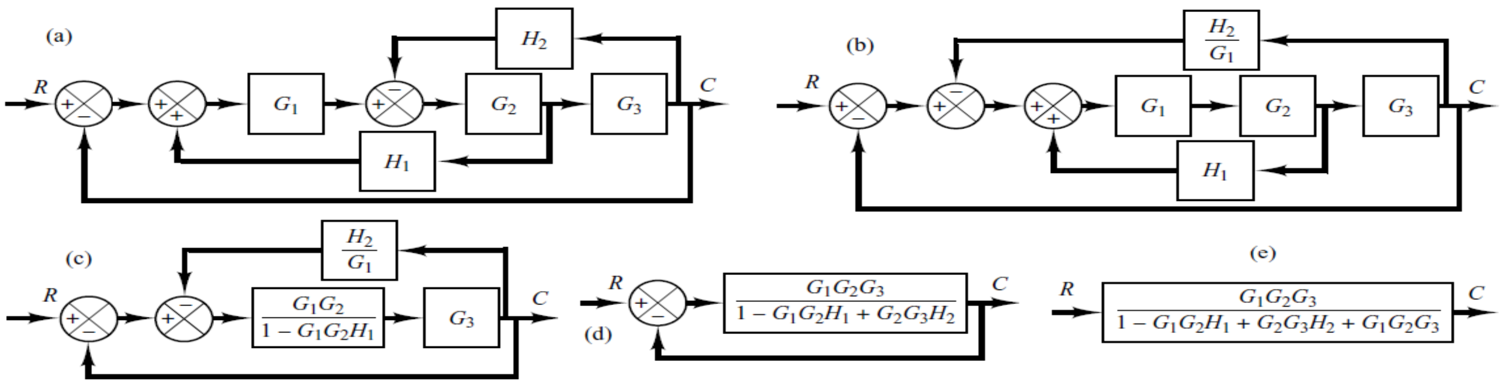


Figure 2–13 (a) Multiple-loop system; (b)–(e) successive reductions of the block diagram shown in (a).

Notice that the numerator of the closed-loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feedforward path. The denominator of $C(s)/R(s)$ is equal to

$$1 + \sum \text{(product of the transfer functions around each loop)}$$
$$= 1 + (-G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3) = 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

(The positive feedback loop yields a negative term in the denominator.)

2–4 MODELING IN STATE SPACE

In this section we shall present introductory material on state-space analysis of control systems.

Modern Control Theory. The modern trend in engineering systems is toward greater complexity, due mainly to the requirements of complex tasks and good accuracy. Complex systems may have multiple inputs and multiple outputs and may be time varying. Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity, and easy access to large scale computers, modern control theory, which is a new approach to the analysis and design of complex control systems, has been developed since around 1960. This new approach is based on the concept of state. The concept of state by itself is not new, since it has been in existence for a long time in the field of classical dynamics and other fields.

Modern Control Theory Versus Conventional Control Theory. Modern control theory is contrasted with conventional control theory in that the former is applicable to multiple-input, multiple-output systems, which may be linear or nonlinear, time invariant or time varying, while the latter is applicable only to linear time-invariant single-input, single-output systems. Also, modern control theory is essentially time-domain approach and frequency domain approach (in certain cases such as H-infinity control), while conventional control theory is a complex frequency-domain approach. Before we proceed further, we must define state, state variables, state vector, and state space.

State. The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Note that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State Variables. The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Note that state variables need not be physically measurable or observable quantities. Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state-space methods. Practically, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

State Vector. If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector $\mathbf{x}$. Such a vector is called a *state vector*. A state vector is thus a vector that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State Space. The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, $\ldots$, x_n axis, where $x_1, x_2, \ldots, x_n$ are state variables, is called a *state space*. Any state can be represented by a point in the state space.

State-Space Equations. In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. As we shall see in Section 2–5, the state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$. Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus the outputs of integrators serve as state variables. The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

Assume that a multiple-input, multiple-output system involves n integrators. Assume also that there are r inputs $u_1(t), u_2(t), \ldots, u_r(t)$ and m outputs $y_1(t), y_2(t), \ldots, y_m(t)$. Define n outputs of the integrators as state variables: $x_1(t), x_2(t), \ldots, x_n(t)$ Then the system may be described by

$$\dot{x}_1(t) = f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \qquad \dot{x}_2(t) = f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)$$
$$\cdots \qquad \dot{x}_n(t) = f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \qquad (2\text{–}8)$$

The outputs $y_1(t), y_2(t), \ldots, y_m(t)$ of the system may be given by

$$y_1(t) = g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \qquad y_2(t) = g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)$$
$$\cdots \qquad y_m(t) = g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \qquad (2\text{–}9)$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

then Equations (2–8) and (2–9) become

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \qquad (2\text{–}10) \qquad\qquad \mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \qquad (2\text{–}11)$$

where Equation (2–10) is the state equation and Equation (2–11) is the output equation. If vector functions $\mathbf{f}$ and/or $\mathbf{g}$ involve time t explicitly, then the system is called a time-varying system.

If Equations (2–10) and (2–11) are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (2\text{–}12) \qquad\qquad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \qquad (2\text{–}13)$$

where $\mathbf{A}(t)$ is called the state matrix, $\mathbf{B}(t)$ the input matrix, $\mathbf{C}(t)$ the output matrix, and $\mathbf{D}(t)$ the direct transmission matrix. (Details of linearization of nonlinear systems about

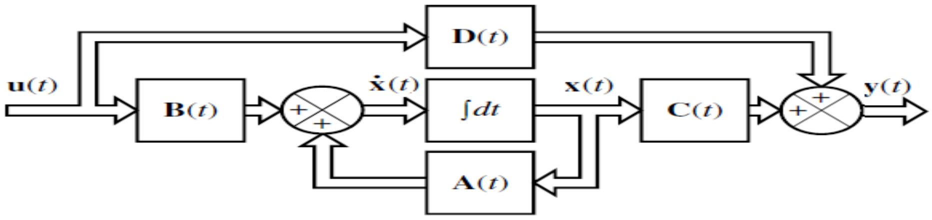


Figure 2–14 Block diagram of the linear, continuous-time control system represented in state space.

the operating state are discussed in Section 2–7.) A block diagram representation of Equations (2–12) and (2–13) is shown in Figure 2–14.

If vector functions **f** and **g** do not involve time t explicitly then the system is called a time-invariant system. In this case, Equations (2–12) and (2–13) can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (2\text{–}14) \qquad\qquad \dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad (2\text{–}15)$$

Equation (2–14) is the state equation of the linear, time-invariant system and Equation (2–15) is the output equation for the same system. In this book we shall be concerned mostly with systems described by Equations (2–14) and (2–15).

In what follows we shall present an example for deriving a state equation and output equation.

EXAMPLE 2–2

Consider the mechanical system shown in Figure 2–15. We assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \qquad (2\text{–}16)$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as $\quad x_1(t) = y(t) \qquad x_2(t) = \dot{y}(t)$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2 \qquad (2\text{–}17)$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \qquad (2\text{–}18)$$

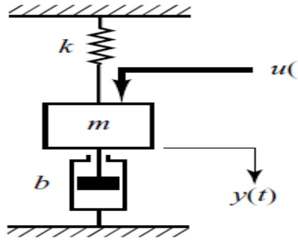


Figure 2–15 Mechanical system.

The output equation is

$$y = x_1 \qquad (2\text{–}19)$$

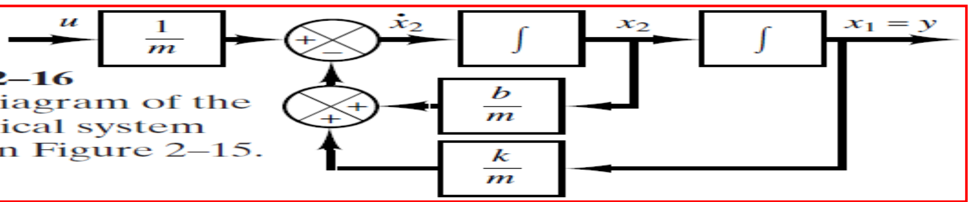


Figure 2–16 Block diagram of the mechanical system shown in Figure 2–15.

In a vector-matrix form, Equations (2–17) and (2–18) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad (2\text{–}20)$$

The output equation, Equation (2–19), can be written as

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2\text{–}21)$$

Equation (2–20) is a state equation and Equation (2–21) is an output equation for the system. They are in the standard form: $\quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad y = \mathbf{C}\mathbf{x} + Du$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0], \qquad D = 0$$

Figure 2–16 is a block diagram for the system. Notice that the outputs of the integrators are state variables.

Correlation Between Transfer Functions and State-Space Equations. In what follows we shall show how to derive the transfer function of a single-input, single-output system from the state-space equations.

Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s) \qquad (2\text{–}22)$$

This system may be represented in state space by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (2\text{–}23) \qquad\qquad y = \mathbf{C}\mathbf{x} + Du \qquad (2\text{–}24)$$

where $\mathbf{x}$ is the state vector, u is the input, and y is the output. The Laplace transforms of Equations (2–23) and (2–24) are given by

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \qquad (2\text{–}25) \qquad\qquad Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s) \qquad (2\text{–}26)$$

Since the transfer function was previously defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions were zero, we set $\mathbf{x}(0)$ in Equation (2–25) to be zero. Then we have

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s) \qquad \text{or} \qquad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

By premultiplying $(s\mathbf{I} - \mathbf{A})^{-1}$ to both sides of this last equation, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) \qquad (2\text{–}27)$$

By substituting Equation (2–27) into Equation (2–26), we get

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D]U(s) \tag{2–28}$$

Upon comparing Equation (2–28) with Equation (2–22), we see that

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \tag{2–29}$$

This is the transfer-function expression of the system in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D.

Note that the right-hand side of Equation (2–29) involves $(s\mathbf{I} - \mathbf{A})^{-1}$. Hence $G(s)$ can be written as

$$G(s) = \frac{Q(s)}{|s\mathbf{I} - \mathbf{A}|}$$

where $Q(s)$ is a polynomial in s. Notice that $|s\mathbf{I} - \mathbf{A}|$ is equal to the characteristic polynomial of $G(s)$. In other words, the eigenvalues of $\mathbf{A}$ are identical to the poles of $G(s)$.

EXAMPLE 2–3

Consider again the mechanical system shown in Figure 2–15. State-space equations for the system are given by Equations (2–20) and (2–21). We shall obtain the transfer function for the system from the state-space equations. By substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D into Equation (2–29), we obtain

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D = [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$G(s) = [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Note that

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

(Refer to Appendix C for the inverse of the 2×2 matrix.) Thus, we have

$$G(s) = [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms^2 + bs + k}$$

which is the transfer function of the system. The same transfer function can be obtained from Equation (2–16).

Transfer Matrix. Next, consider a multiple-input, multiple-output system. Assume that there are r inputs $u_1, u_2, \ldots, u_r$, and m outputs $y_1, y_2, \ldots, y_m$. Define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

The transfer matrix $\mathbf{G}(s)$ relates the output $\mathbf{Y}(s)$ to the input $\mathbf{U}(s)$, or $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$ where $\mathbf{G}(s)$ is given by $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

[The derivation for this equation is the same as that for Equation (2–29).] Since the input vector $\mathbf{u}$ is r dimensional and the output vector $\mathbf{y}$ is m dimensional, the transfer matrix $\mathbf{G}(s)$ is an $m \times r$ matrix.

2–5 STATE-SPACE REPRESENTATION OF SCALAR DIFFERENTIAL EQUATION SYSTEMS

A dynamic system consisting of a finite number of lumped elements may be described by ordinary differential equations in which time is the independent variable. By use of vector-matrix notation, an nth-order differential equation may be expressed by a first-order vector-matrix differential equation. If n elements of the vector are a set of state variables, then the vector-matrix differential equation is a *state* equation. In this section we shall present methods for obtaining state-space representations of continuous-time systems.

State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms.

Consider the following nth-order system:

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{2–30}$$

Noting that the knowledge of $y(0), \dot{y}(0), \ldots, \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of n state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher-order derivative terms are inaccurate, due to the noise effects inherent in any practical situations, such a choice of the state variables may not be desirable.) Let us define

$$x_1 = y \qquad x_2 = \dot{y} \qquad \cdot \quad \cdot \quad \cdot \qquad x_n = \overset{(n-1)}{y}$$

Then Equation (2–30) can be written as

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \cdot \quad \cdot \quad \cdot \qquad \dot{x}_{n-1} = x_n \qquad \dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{2–31}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \qquad (2\text{–}32)$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

[Note that D in Equation (2–24) is zero.] The first-order differential equation, Equation (2–31), is the state equation, and the algebraic equation, Equation (2–32), is the output equation. Note that the state-space representation for the transfer function system $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$ is given also by Equations (2–31) and (2–32).

State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms. Consider the differential equation system that involves derivatives of the forcing function, such as

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \qquad (2\text{–}33)$$

The main problem in defining the state variables for this case lies in the derivative terms of the input u. The state variables must be such that they will eliminate the derivatives of u in the state equation.

One way to obtain a state equation and output equation for this case is to define the following n variables as a set of n state variables:

$$x_1 = y - \beta_0 u \qquad x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \qquad x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\cdots \quad x_n = \overset{(n-1)}{y} - \beta_0\overset{(n-1)}{u} - \beta_1\overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u \qquad (2\text{–}34)$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are determined from

$$\beta_0 = b_0 \qquad \beta_1 = b_1 - a_1\beta_0 \qquad \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \qquad \beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

$$\cdots \quad \beta_{n-1} = b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0 \qquad (2\text{–}35)$$

With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables.) With the present choice of state variables, we obtain

$$\dot{x}_1 = x_2 + \beta_1 u \qquad \dot{x}_2 = x_3 + \beta_2 u \qquad \cdots \qquad \dot{x}_{n-1} = x_n + \beta_{n-1}u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + \beta_n u \qquad (2\text{–}36)$$

where β_n is given by $\quad \beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$

[To derive Equation (2–36), see Problem **A–2–6**.] In terms of vector-matrix equations, Equation (2–36) and the output equation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u \qquad y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (2\text{–}37) \qquad\qquad y = \mathbf{C}\mathbf{x} + Du \qquad (2\text{–}38)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \quad D = \beta_0 = b_0$$

In this state-space representation, matrices **A** and **C** are exactly the same as those for the system of Equation (2–30). The derivatives on the right-hand side of Equation (2–33) affect only the elements of the **B** matrix.

Note that the state-space representation for the transfer function $\dfrac{Y(s)}{U(s)} = \dfrac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$ is given also by Equations (2–37) and (2–38).

There are many ways to obtain state-space representations of systems. Methods for obtaining canonical representations of systems in state space (such as controllable canonical form, observable canonical form, diagonal canonical form, and Jordan canonical form) are presented in Chapter 9.

MATLAB can also be used to obtain state-space representations of systems from transfer-function representations, and vice versa. This subject is presented in Section 2–6.

TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful to transform the system model from transfer function to state space, and vice versa. We shall begin our discussion with transformation from transfer function to state space.

Let us write the closed-loop transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

[A,B,C,D] = tf2ss(num,den)

will give a state-space representation. It is important to note that the state-space representation for any system is not unique. There are many (infinitely many) state-space representations for the same system. The MATLAB command gives one possible such state-space representation.

Transformation from Transfer Function to State Space Representation. Consider the transfer-function system

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+10)(s^2+4s+16)} = \frac{s}{s^3+14s^2+56s+160} \qquad (2\text{–}39)$$

There are many (infinitely many) possible state-space representations for this system. One possible state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Another possible state-space representation (among infinitely many alternatives) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad (2\text{–}40)$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \qquad (2\text{–}41)$$

MATLAB transforms the transfer function given by Equation (2–39) into the state-space representation given by Equations (2–40) and (2–41). For the example system considered here, MATLAB Program 2–2 will produce matrices **A**, **B**, **C**, and *D*.

MATLAB Program 2–2

```
num = [1      0];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =
   -14   -56   -160
     1     0      0
     0     1      0

B =
     1
     0
     0

C =   0    1    0

D = 0
```

Transformation from State Space Representation to Transfer Function. To obtain the transfer function from state-space equations, use the following command:

[num,den] = ss2tf(A,B,C,D,iu)

iu must be specified for systems with more than one input. For example, if the system has three inputs ($u1$, $u2$, $u3$), then iu must be either 1, 2, or 3, where 1 implies $u1$, 2 implies $u2$, and 3 implies $u3$.

If the system has only one input, then either [num,den] = ss2tf(A,B,C,D)

or [num,den] = ss2tf(A,B,C,D,1)

may be used. For the case where the system has multiple inputs and multiple outputs, see Problem **A–2–12**.

EXAMPLE 2–4

Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATLAB Program 2-3 will produce the transfer function for the given system. The transfer function obtained is given by

$$\frac{Y(s)}{U(s)} = \frac{25s+5}{s^3+5s^2+25s+5}$$

MATLAB Program 2–3

```
A = [0   1   0;   0   0   1;   -5   -25   -5];
B = [0; 25; -120];
C = [1   0    0];
D = [0];
[num,den] = ss2tf(A,B,C,D)
num = 0   0.0000   25.0000   5.0000
den  1.0000   5.0000   25.0000  5.0000
% ***** The same result can be obtained by
entering the following command: *****
[num,den] = ss2tf(A,B,C,D,1)
num = 0   0.0000   25.0000  5.0000
den =1.0000   5.0000   25.0000  5.0000
```

2–7 LINEARIZATION OF NONLINEAR MATHEMATICAL MODELS

Nonlinear Systems. A system is nonlinear if the principle of superposition does not apply. Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results.

Although many physical relationships are often represented by linear equations, in most cases actual relationships are not quite linear. In fact, a careful study of physical systems reveals that even so-called "linear systems" are really linear only in limited operating ranges. In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve nonlinear relationships among the variables. For example, the output of a component may saturate for large input signals. There may be a dead space that affects small signals. (The dead space of a component is a small range of input variations to which the component is insensitive.) Square-law nonlinearity may occur in some components. For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity.

Linearization of Nonlinear Systems. In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around the equilibrium. (It should be pointed out that there are many exceptions to such a case.) However, if the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system. Such a linear system is equivalent to the nonlinear system considered within a limited operating range. Such a linearized model (linear, time-invariant model) is very important in control engineering.

The linearization procedure to be presented in the following is based on the expansion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term. Because we neglect higher-order terms of the Taylor series expansion, these neglected terms must be small enough; that is, the variables deviate only slightly from the operating condition. (Otherwise, the result will be inaccurate.)

Linear Approximation of Nonlinear Mathematical Models. To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition. Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x) \tag{2–42}$$

If the normal operating condition corresponds to $\bar{x}, \bar{y}$, then Equation (2–42) may be expanded into a Taylor series about this point as follows:

$$y = f(x) = f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x - \bar{x})^2 + \cdots \tag{2–43}$$

where the derivatives $df/dx, d^2f/dx^2, \ldots$ are evaluated at $x = \bar{x}$. If the variation $x - \bar{x}$ is small, we may neglect the higher-order terms in $x - \bar{x}$. Then Equation (2–43) may be written as

$$y = \bar{y} + K(x - \bar{x}) \tag{2–44}$$

where

$$\bar{y} = f(\bar{x}) \qquad K = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

Equation (2–44) may be rewritten as $y - \bar{y} = K(x - \bar{x})$ (2–45)

which indicates that $y - \bar{y}$ is proportional to $x - \bar{x}$. Equation (2–45) gives a linear mathematical model for the nonlinear system given by Equation (2–42) near the operating point $x = \bar{x}, y = \bar{y}$.

Next, consider a nonlinear system whose output y is a function of two inputs x_1 and x_2, so that

$$y = f(x_1, x_2) \tag{2–46}$$

To obtain a linear approximation to this nonlinear system, we may expand Equation (2–46) into a Taylor series about the normal operating point $\bar{x}_1, \bar{x}_2$. Then Equation (2–46) becomes

$$y = f(\bar{x}_1, \bar{x}_2) + \left[\frac{\partial f}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f}{\partial x_2}(x_2 - \bar{x}_2)\right]$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - \bar{x}_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - \bar{x}_2)^2\right] + \cdots$$

where the partial derivatives are evaluated at $x_1 = \bar{x}_1, x_2 = \bar{x}_2$. Near the normal operating point, the higher-order terms may be neglected. The linear mathematical model of this nonlinear system in the neighborhood of the normal operating condition is then given by

$$y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_2)$$

where

$$\bar{y} = f(\bar{x}_1, \bar{x}_2) \qquad K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2} \qquad K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2}$$

The linearization technique presented here is valid in the vicinity of the operating condition. If the operating conditions vary widely, however, such linearized equations are not adequate, and nonlinear equations must be dealt with. It is important to remember that a particular mathematical model used in analysis and design may accurately represent the dynamics of an actual system for certain operating conditions, but may not be accurate for other operating conditions.

EXAMPLE 2–5 Linearize the nonlinear equation $z = xy$

in the region $5 \le x \le 7, 10 \le y \le 12$. Find the error if the linearized equation is used to calculate the value of z when $x = 5, y = 10$.

Since the region considered is given by $5 \le x \le 7, 10 \le y \le 12$, choose $\bar{x} = 6, \bar{y} = 11$. Then $\bar{z} = \bar{x}\bar{y} = 66$. Let us obtain a linearized equation for the nonlinear equation near a point $\bar{x} = 6$, $\bar{y} = 11$.

Expanding the nonlinear equation into a Taylor series about point $x = \bar{x}, y = \bar{y}$ and neglecting the higher-order terms, we have

$$z - \bar{z} = a(x - \bar{x}) + b(y - \bar{y})$$

where

$$a = \left.\frac{\partial(xy)}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{y} = 11 \qquad b = \left.\frac{\partial(xy)}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{x} = 6$$

Hence the linearized equation is

$$z - 66 = 11(x - 6) + 6(y - 11) \qquad \text{or} \qquad z = 11x + 6y - 66$$

When $x = 5, y = 10$, the value of z given by the linearized equation is

$$z = 11x + 6y - 66 = 55 + 60 - 66 = 49$$

The exact value of z is $z = xy = 50$. The error is thus $50 - 49 = 1$. In terms of percentage, the error is 2%.

EXAMPLE PROBLEMS AND SOLUTIONS

A–2–1. Simplify the block diagram shown in Figure 2–17.

Solution. First, move the branch point of the path involving H_1 outside the loop involving H_2, as shown in Figure 2–18(a). Then eliminating two loops results in Figure 2–18(b). Combining two blocks into one gives Figure 2–18(c).

A–2–2. Simplify the block diagram shown in Figure 2–19. Obtain the transfer function relating $C(s)$ and $R(s)$.

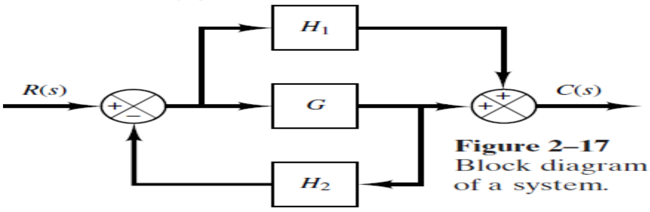
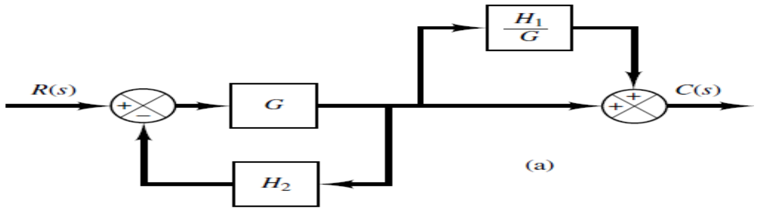


Figure 2–17 Block diagram of a system.



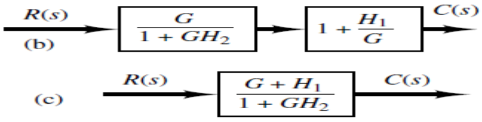


Figure 2–18 Simplified block diagrams for the system shown in Figure 2–17.

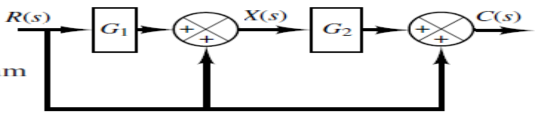


Figure 2–19 Block diagram of a system.

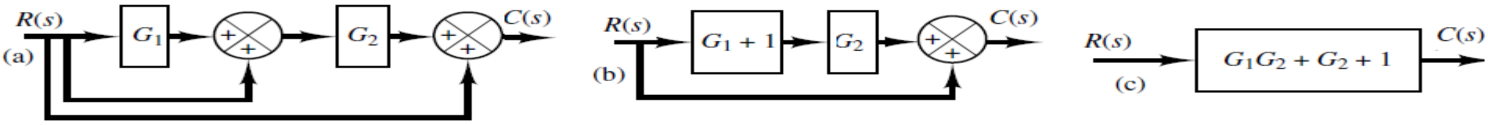


Figure 2–20 Reduction of the block diagram shown in Figure 2–19.

Solution. The block diagram of Figure 2–19 can be modified to that shown in Figure 2–20(a). Eliminating the minor feedforward path, we obtain Figure 2–20(b), which can be simplified to Figure 2–20(c). The transfer function $C(s)/R(s)$ is thus given by

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

The same result can also be obtained by proceeding as follows: Since signal $X(s)$ is the sum of two signals $G_1R(s)$ and $R(s)$, we have

$$X(s) = G_1R(s) + R(s)$$

The output signal $C(s)$ is the sum of $G_2X(s)$ and $R(s)$. Hence

$$C(s) = G_2X(s) + R(s) = G_2[G_1R(s) + R(s)] + R(s)$$

And so we have the same result as before:

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

A–2–3. Simplify the block diagram shown in Figure 2–21. Then obtain the closed-loop transfer function $C(s)/R(s)$.

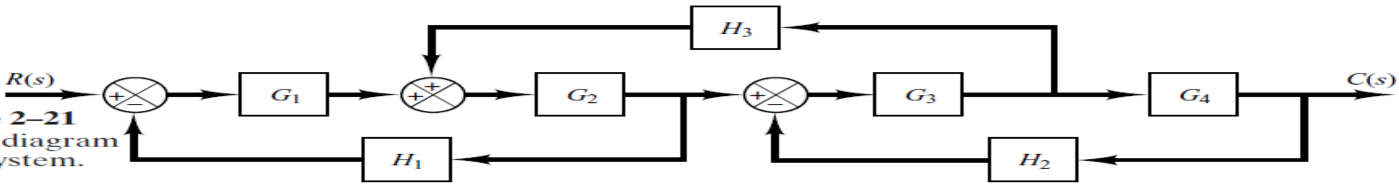
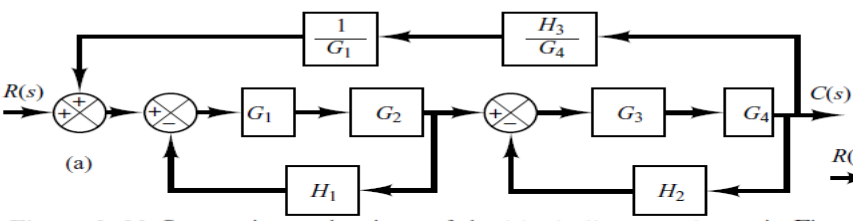


Figure 2–21 Block diagram of a system.



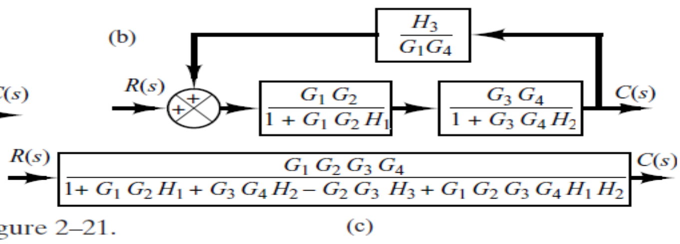


Figure 2–22 Successive reductions of the block diagram shown in Figure 2–21.

Solution. First move the branch point between G_3 and G_4 to the right-hand side of the loop containing G_3, G_4, and H_2. Then move the summing point between G_1 and G_2 to the left-hand side of the first summing point. See Figure 2–22(a). By simplifying each loop, the block diagram can be modified as shown in Figure 2–22(b). Further simplification results in Figure 2–22(c), from which the closed-loop transfer function $C(s)/R(s)$ is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2H_1 + G_3G_4H_2 - G_2G_3H_3 + G_1G_2G_3G_4H_1H_2}$$

A–2–4. Obtain transfer functions $C(s)/R(s)$ and $C(s)/D(s)$ of the system shown in Figure 2–23.

Solution. From Figure 2–23 we have $U(s) = G_fR(s) + G_cE(s)$ (2–47)

$C(s) = G_p[D(s) + G_1U(s)]$ (2–48) $E(s) = R(s) - HC(s)$ (2–49)

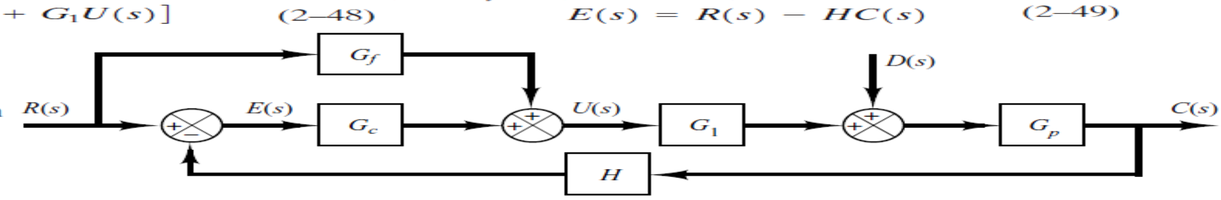


Figure 2–23 Control system with reference input and disturbance input.

By substituting Equation (2–47) into Equation (2–48), we get

$$C(s) = G_p D(s) + G_1 G_p [G_f R(s) + G_c E(s)] \tag{2–50}$$

By substituting Equation (2–49) into Equation (2–50), we obtain

$$C(s) = G_p D(s) + G_1 G_p \{G_f R(s) + G_c [R(s) - HC(s)]\}$$

Solving this last equation for $C(s)$, we get

$$C(s) + G_1 G_p G_c H C(s) = G_p D(s) + G_1 G_p (G_f + G_c) R(s)$$

Hence

$$C(s) = \frac{G_p D(s) + G_1 G_p (G_f + G_c) R(s)}{1 + G_1 G_p G_c H} \tag{2–51}$$

Note that Equation (2–51) gives the response $C(s)$ when both reference input $R(s)$ and disturbance input $D(s)$ are present.

To find transfer function $C(s)/R(s)$, we let $D(s) = 0$ in Equation (2–51). Then we obtain

$$\frac{C(s)}{R(s)} = \frac{G_1 G_p (G_f + G_c)}{1 + G_1 G_p G_c H}$$

Similarly, to obtain transfer function $C(s)/D(s)$, we let $R(s) = 0$ in Equation (2–51). Then $C(s)/D(s)$ can be given by

$$\frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1 G_p G_c H}$$

A–2–5. Figure 2–24 shows a system with two inputs and two outputs. Derive $C_1(s)/R_1(s)$, $C_1(s)/R_2(s)$, $C_2(s)/R_1(s)$, and $C_2(s)/R_2(s)$. (In deriving outputs for $R_1(s)$, assume that $R_2(s)$ is zero, and vice versa.)

Solution. From the figure,

we obtain $C_1 = G_1(R_1 - G_3 C_2)$ (2–52)

$$C_2 = G_4(R_2 - G_2 C_1) \tag{2–53}$$

By substituting Equation (2–53) into Equation (2–52), we obtain

$$C_1 = G_1[R_1 - G_3 G_4 (R_2 - G_2 C_1)] \tag{2–54}$$

By substituting Equation (2–52) into Equation (2–53), we get

$$C_2 = G_4[R_2 - G_2 G_1 (R_1 - G_3 C_2)] \tag{2–55}$$

Solving Equation (2–54) for C_1, we obtain

$$C_1 = \frac{G_1 R_1 - G_1 G_3 G_4 R_2}{1 - G_1 G_2 G_3 G_4} \tag{2–56}$$

Solving Equation (2–55) for C_2 gives

$$C_2 = \frac{-G_1 G_2 G_4 R_1 + G_4 R_2}{1 - G_1 G_2 G_3 G_4} \tag{2–57}$$

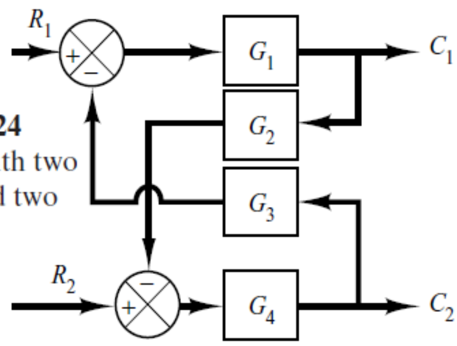


Figure 2–24 System with two inputs and two outputs.

Equations (2–56) and (2–57) can be combined in the form of the transfer matrix as follows:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \dfrac{G_1}{1 - G_1 G_2 G_3 G_4} & -\dfrac{G_1 G_3 G_4}{1 - G_1 G_2 G_3 G_4} \\ -\dfrac{G_1 G_2 G_4}{1 - G_1 G_2 G_3 G_4} & \dfrac{G_4}{1 - G_1 G_2 G_3 G_4} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Then the transfer functions $C_1(s)/R_1(s)$, $C_1(s)/R_2(s)$, $C_2(s)/R_1(s)$ and $C_2(s)/R_2(s)$ can be obtained as follows:

$$\frac{C_1(s)}{R_1(s)} = \frac{G_1}{1 - G_1 G_2 G_3 G_4}, \qquad \frac{C_1(s)}{R_2(s)} = -\frac{G_1 G_3 G_4}{1 - G_1 G_2 G_3 G_4}$$

$$\frac{C_2(s)}{R_1(s)} = -\frac{G_1 G_2 G_4}{1 - G_1 G_2 G_3 G_4}, \qquad \frac{C_2(s)}{R_2(s)} = \frac{G_4}{1 - G_1 G_2 G_3 G_4}$$

Note that Equations (2–56) and (2–57) give responses C_1 and C_2, respectively, when both inputs R_1 and R_2 are present.

Notice that when $R_2(s) = 0$, the original block diagram can be simplified to those shown in Figures 2–25(a) and (b). Similarly, when $R_1(s) = 0$, the original block diagram can be simplified to those shown in Figures 2–25(c) and (d). From these simplified block diagrams we can also obtain $C_1(s)/R_1(s)$, $C_2(s)/R_1(s)$, $C_1(s)/R_2(s)$, and $C_2(s)/R_2(s)$, as shown to the right of each corresponding block diagram.

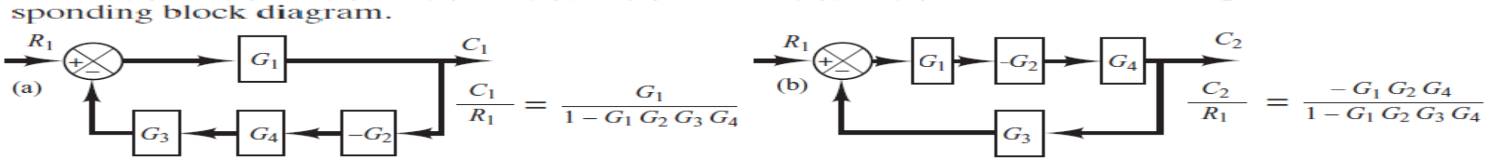
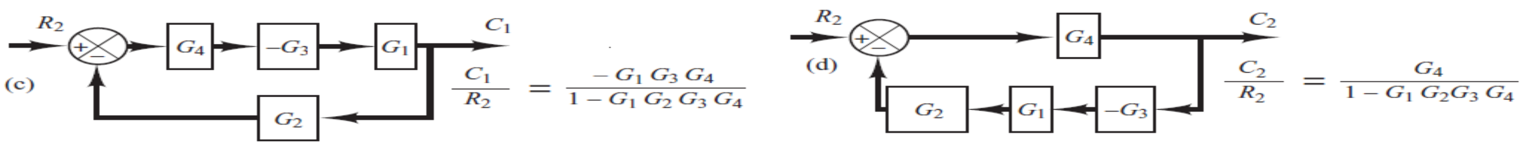


Figure 2–25 Simplified block diagrams and corresponding closed-loop transfer functions.



A–2–6. Show that for the differential equation system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

state and output equations can be given, respectively, by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_3 & -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}u \quad \text{and} \quad y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \beta_0 u$$

where state variables are defined by $x_1 = y - \beta_0 u$

$$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \qquad x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

and $\beta_0 = b_0 \qquad \beta_1 = b_1 - a_1\beta_0 \qquad \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \qquad \beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$

Solution. From the definition of state variables x_2 and x_3, we have

$$\dot{x}_1 = x_2 + \beta_1 u \quad (2\text{–}61) \qquad \dot{x}_2 = x_3 + \beta_2 u \quad (2\text{–}62)$$

To derive the equation for $\dot{x}_3$, we first note from Equation (2–58) that

$$\dddot{y} = -a_1\ddot{y} - a_2\dot{y} - a_3 y + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Since $x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u$ we have $\dot{x}_3 = \dddot{y} - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u}$

$$\dot{x}_3 = (-a_1\ddot{y} - a_2\dot{y} - a_3 y) + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u}$$

$$\dot{x}_3 = -a_1(\ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u) - a_1\beta_0\ddot{u} - a_1\beta_1\dot{u} - a_1\beta_2 u - a_2(\dot{y} - \beta_0\dot{u} - \beta_1 u) - a_2\beta_0\dot{u}$$
$$- a_2\beta_1 u - a_3(y - \beta_0 u) - a_3\beta_0 u + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u}$$

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_0 - \beta_0)\dddot{u} + (b_1 - \beta_1 - a_1\beta_0)\ddot{u}$$
$$+ (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)\dot{u} + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u$$

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u = -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u$$

Hence, we get

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u \qquad (2\text{–}63)$$

Combining Equations (2–61), (2–62), and (2–63) into a vector-matrix equation, we obtain Equation (2–59). Also, from the definition of state variable x_1, we get the output equation given by Equation (2–60).

A–2–7. Obtain a state-space equation and output equation for the system defined by

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s + 2}{s^3 + 4s^2 + 5s + 2}$$

Solution. From the given transfer function, the differential equation for the system is

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\dddot{u} + \ddot{u} + \dot{u} + 2u$$

Comparing this equation with the standard equation given by Equation (2–33), rewritten

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we find $a_1 = 4, \quad a_2 = 5, \quad a_3 = 2 \quad b_0 = 2, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 2$

Referring to Equation (2–35), we have $\beta_0 = b_0 = 2 \quad \beta_1 = b_1 - a_1\beta_0 = -7$

$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 19 \qquad \beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = -43$

Referring to Equation (2–34), we define

$x_1 = y - \beta_0 u = y - 2u \qquad x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 + 7u \qquad x_3 = \dot{x}_2 - \beta_2 u = \dot{x}_2 - 19u$

Then referring to Equation (2–36), $\dot{x}_1 = x_2 - 7u \qquad \dot{x}_2 = x_3 + 19u$

$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u = -2x_1 - 5x_2 - 4x_3 - 43u$

Hence, the state-space representation of the system is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -5 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}-7\\ 19\\ -43\end{bmatrix}u \qquad y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + 2u$$

This is one possible state-space representation of the system. There are many (infinitely many) others. If we use MATLAB, it produces the following state-space representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-4 & -5 & -2\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u \qquad y = [-7 \quad -9 \quad -2]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + 2u$$

See MATLAB Program 2-4. (Note that all state-space representations for the same system are equivalent.)

MATLAB Program 2–4

```
num = [2  1  1  2];
den = [1  4  5  2];
[A,B,C,D] = tf2ss(num,den)
```

```
     -4  -5  -2        1     C = -7  -9  -2
A =   1   0   0   B =  0     D = 2
      0   1   0        0
```

A–2–8. Obtain a state-space model of the system shown in Figure 2–26.

Solution. The system involves one integrator and two delayed integrators. The output of each integrator or delayed integrator can be a state variable. Let us define the output of the plant as x_1, the output of the controller as x_2, and the output of the sensor as x_3. Then we obtain

$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s + 5} \qquad \frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s} \qquad \frac{X_3(s)}{X_1(s)} = \frac{1}{s + 1} \qquad Y(s) = X_1(s)$$

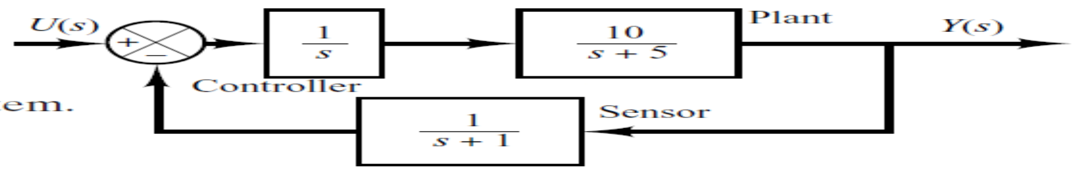


Figure 2–26
Control system.

which can be rewritten as $sX_1(s) = -5X_1(s) + 10X_2(s)$

$sX_2(s) = -X_3(s) + U(s) \qquad sX_3(s) = X_1(s) - X_3(s) \qquad Y(s) = X_1(s)$

By taking the inverse Laplace transforms of the preceding four equations, we obtain

$\dot{x}_1 = -5x_1 + 10x_2 \qquad \dot{x}_2 = -x_3 + u \qquad \dot{x}_3 = x_1 - x_3 \qquad y = x_1$

Thus, a state-space model of the system in the standard form is given by

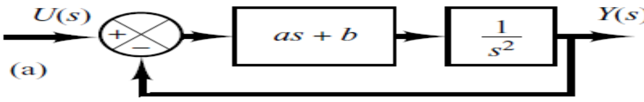
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-5 & 10 & 0\\ 0 & 0 & -1\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u \qquad y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

It is important to note that this is not the only state-space representation of the system. Infinitely many other state-space representations are possible. However, the number of state variables is the same in any state-space representation of the same system. In the present system, the number of state variables is three, regardless of what variables are chosen as state variables.

A–2–9. Obtain a state-space model for the system shown in Figure 2–27(a).

Solution. First, notice that $(as + b)/s^2$ involves a derivative term. Such a derivative term may be avoided if we modify $(as + b)/s^2$ as
$$\frac{as + b}{s^2} = \left(a + \frac{b}{s}\right)\frac{1}{s}$$
Using this modification, the block diagram of Figure 2–27(a) can be modified to that shown in Figure 2–27(b).

Define the outputs of the integrators as state variables, as shown in Figure 2–27(b). Then from Figure 2–27(b) we obtain
$$\frac{X_1(s)}{X_2(s) + a[U(s) - X_1(s)]} = \frac{1}{s} \qquad \frac{X_2(s)}{U(s) - X_1(s)} = \frac{b}{s} \qquad Y(s) = X_1(s)$$ which may be modified to
$$sX_1(s) = X_2(s) + a[U(s) - X_1(s)] \qquad sX_2(s) = -bX_1(s) + bU(s) \qquad Y(s) = X_1(s)$$



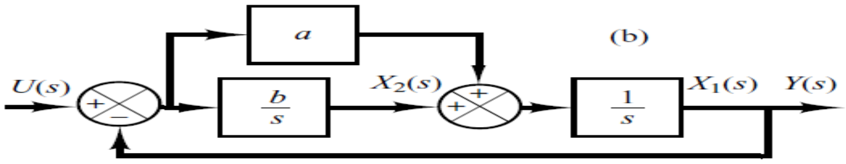


Figure 2–27 (a) Control system; (b) modified block diagram.

Taking the inverse Laplace transforms of the preceding three equations, we obtain
$$\dot{x}_1 = -ax_1 + x_2 + au \qquad \dot{x}_2 = -bx_1 + bu \qquad y = x_1$$
Rewriting the state and output equations in the standard vector-matrix form, we obtain
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}u \qquad y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A–2–10. Obtain a state-space representation of the system shown in Figure 2–28(a).

Solution. In this problem, first expand $(s + z)/(s + p)$ into partial fractions.
$$\frac{s + z}{s + p} = 1 + \frac{z - p}{s + p}$$
Next, convert $K/[s(s + a)]$ into the product of K/s and $1/(s + a)$. Then redraw the block diagram, as shown in Figure 2–28(b). Defining a set of state variables, as shown in Figure 2–28(b), we obtain the following equations: $\dot{x}_1 = -ax_1 + x_2$
$$\dot{x}_2 = -Kx_1 + Kx_3 + Ku \qquad \dot{x}_3 = -(z - p)x_1 - px_3 + (z - p)u \qquad y = x_1$$

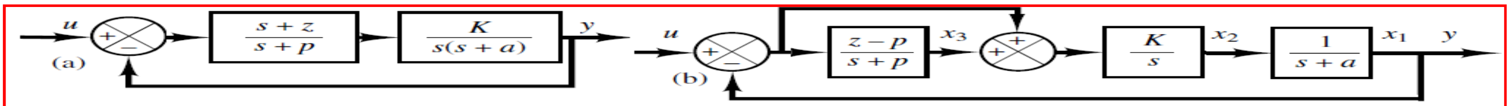


Figure 2–28 (a) Control system; (b) block diagram defining state variables for the system.

Rewriting gives
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a & 1 & 0 \\ -K & 0 & K \\ -(z - p) & 0 & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ z - p \end{bmatrix}u \qquad y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
Notice that the output of the integrator and the outputs of the first-order delayed integrators $[1/(s + a)$ and $(z - p)/(s + p)]$ are chosen as state variables. It is important to remember that the output of the block $(s + z)/(s + p)$ in Figure 2–28(a) cannot be a state variable, because this block involves a derivative term, $s + z$.

A–2–11. Obtain the transfer function of the system defined by
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \qquad y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
Solution. Referring to Equation (2–29), the transfer function $G(s)$ is given by
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$
In this problem, matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D are
$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0], \qquad D = 0$$
Hence
$$G(s) = [1 \quad 0 \quad 0]\begin{bmatrix} s + 1 & -1 & 0 \\ 0 & s + 1 & -1 \\ 0 & 0 & s + 2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= [1 \quad 0 \quad 0]\begin{bmatrix} \frac{1}{s + 1} & \frac{1}{(s + 1)^2} & \frac{1}{(s + 1)^2(s + 2)} \\ 0 & \frac{1}{s + 1} & \frac{1}{(s + 1)(s + 2)} \\ 0 & 0 & \frac{1}{s + 2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{(s + 1)^2(s + 2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

A–2–12. Linearize the nonlinear equation $z = x^2 + 4xy + 6y^2$

in the region defined by $8 \le x \le 10$, $2 \le y \le 4$.

Solution. Define
$$f(x, y) = z = x^2 + 4xy + 6y^2$$
Then
$$z = f(x, y) = f(\bar{x}, \bar{y}) + \left[\frac{\partial f}{\partial x}(x - \bar{x}) + \frac{\partial f}{\partial y}(y - \bar{y})\right]_{x=\bar{x},\, y=\bar{y}} + \cdots$$
where we choose $\bar{x} = 9$, $\bar{y} = 3$.

Since the higher-order terms in the expanded equation are small, neglecting these higher-order terms, we obtain
$$z - \bar{z} = K_1(x - \bar{x}) + K_2(y - \bar{y})$$
where
$$K_1 = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = 2\bar{x} + 4\bar{y} = 2 \times 9 + 4 \times 3 = 30$$
$$K_2 = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = 4\bar{x} + 12\bar{y} = 4 \times 9 + 12 \times 3 = 72$$
$$\bar{z} = \bar{x}^2 + 4\bar{x}\bar{y} + 6\bar{y}^2 = 9^2 + 4 \times 9 \times 3 + 6 \times 9 = 243$$
Thus
$$z - 243 = 30(x - 9) + 72(y - 3)$$
Hence a linear approximation of the given nonlinear equation near the operating point is
$$z - 30x - 72y + 243 = 0$$

Mathematical Modeling of Mechanical Systems and Electrical Systems

3

3–1 INTRODUCTION

This chapter presents mathematical modeling of mechanical systems and electrical systems. In Chapter 2 we obtained mathematical models of a simple electrical circuit and a simple mechanical system. In this chapter we consider mathematical modeling of a variety of mechanical systems and electrical systems that may appear in control systems.

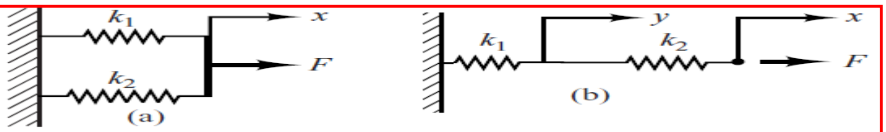
The fundamental law govering mechanical systems is Newton's second law. In Section 3–2 we apply this law to various mechanical systems and derive transfer-function models and state-space models.

The basic laws governing electrical circuits are Kirchhoff's laws. In Section 3–3 we obtain transfer-function models and state-space models of various electrical circuits and operational amplifier systems that may appear in many control systems.

3–2 MATHEMATICAL MODELING OF MECHANICAL SYSTEMS

This section first discusses simple spring systems and simple damper systems. Then we derive transfer-function models and state-space models of various mechanical systems.

Figure 3–1
(a) System consisting of two springs in parallel; (b) system consisting of two springs in series.



EXAMPLE 3–1

Let us obtain the equivalent spring constants for the systems shown in Figures 3–1(a) and (b), respectively.

For the springs in parallel [Figure 3–1(a)] the equivalent spring constant k_{eq} is obtained from $k_1 x + k_2 x = F = k_{eq} x$ or $k_{eq} = k_1 + k_2$

For the springs in series [Figure–3–1(b)], the force in each spring is the same. Thus

$$k_1 y = F, \qquad k_2(x - y) = F$$

Elimination of y from these two equations results in $k_2\left(x - \frac{F}{k_1}\right) = F$

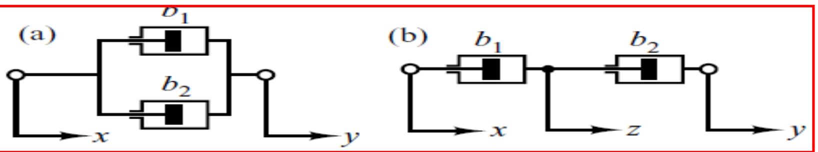
or $k_2 x = F + \frac{k_2}{k_1}F = \frac{k_1 + k_2}{k_1}F$

equivalent spring constant k_{eq} for this case is then found as $k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = 1 / \frac{1}{k_1} + \frac{1}{k_2}$

EXAMPLE 3–2

Let us obtain the equivalent viscous-friction coefficient b_{eq} for each of the damper systems shown in Figures 3–2(a) and (b). An oil-filled damper is often called a dashpot. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy.

Figure 3–2
(a) Two dampers connected in parallel; (b) two dampers connected in series.



(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$f = b_{eq}(\dot{y} - \dot{x})$ Hence $b_{eq} = b_1 + b_2$

(b) The force f due to the dampers is $f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z})$ (3–1)

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (3–1), we have

$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$ or $\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})$ (3–2)

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (3–2) into Equation (3–1), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2\left[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})\right] = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Thus, $f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$ Hence, $b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = 1 / \frac{1}{b_1} + \frac{1}{b_2}$

EXAMPLE 3–3

Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3–3. Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u}$ = constant. The displacement $y(t)$ of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u) \qquad \text{or} \qquad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

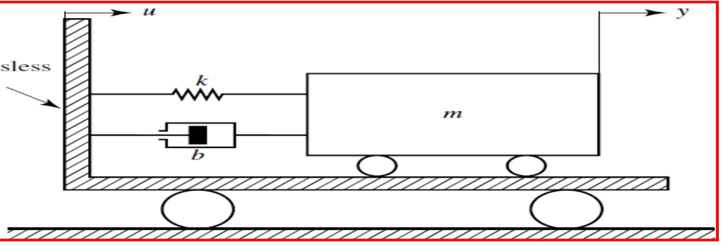
$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

Figure 3–3 Spring-mass-dashpot system mounted on a cart.



Next we shall obtain a state-space model of this system. We shall first compare the differential equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form $\quad \ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

Referring to Equation (3–35), we have $\beta_0 = b_0 = 0$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m} \qquad \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then, referring to Equation (2–34), define

$$x_1 = y - \beta_0 u = y \qquad x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

From Equation (2–36) we have $\quad \dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$

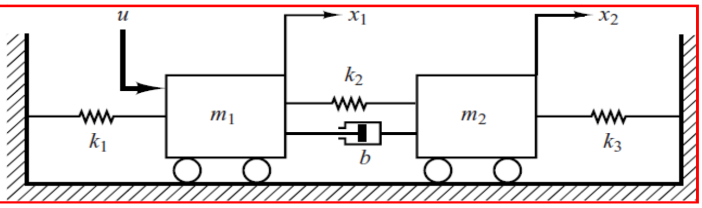
$$\dot{x}_2 = -a_2x_1 - a_1x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes $\quad y = x_1$

or $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{b}{m}\\ \frac{k}{m} - \left(\frac{b}{m}\right)^2\end{bmatrix}u \qquad (3\text{–}3) \qquad \text{and} \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad (3\text{–}4)$$

Equations (3–3) and (3–4) give a state-space representation of the system. (Note that this is not the only state-space representation. There are infinitely many state-space representations for the system.)

Figure 3–4 Mechanical system.



EXAMPLE 3–4

Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–4. The equations of motion for the system shown in Figure 3–4 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u \qquad m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u \qquad m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1s^2 + bs + (k_1 + k_2)]X_1(s) = (bs + k_2)X_2(s) + U(s) \qquad (3\text{–}5)$$

$$[m_2s^2 + bs + (k_2 + k_3)]X_2(s) = (bs + k_2)X_1(s) \qquad (3\text{–}6)$$

Solving Equation (3–6) for $X_2(s)$ and substituting it into Equation (3–5) and simplifying, we get

$$[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]X_1(s) = (m_2s^2 + bs + k_2 + k_3)U(s)$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3\text{–}7)$$

From Equations (3–6) and (3–7) we have

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3\text{–}8)$$

Equations (3–7) and (3–8) are the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, respectively.

EXAMPLE 3–5

An inverted pendulum mounted on a motor-driven cart is shown in Figure 3–5(a). This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider

only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then $\quad x_G = x + l\sin\theta \qquad y_G = l\cos\theta$

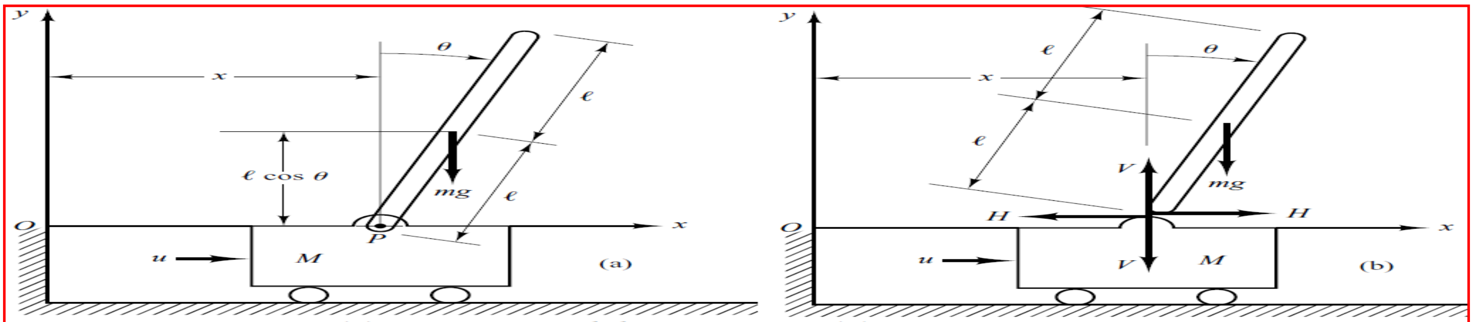


Figure 3–5 (a) Inverted pendulum system; (b) free-body diagram.

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–5(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{3–9}$$

where I is the moment of inertia of the rod about its center of gravity.

The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{3–10}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{3–11}$$

The horizontal motion of cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{3–12}$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that $\sin\theta \doteqdot \theta$, $\cos\theta = 1$, and $\theta\dot{\theta}^2 = 0$. Then, Equations (3–9) through (3–11) can be linearized. The linearized equations are

$I\ddot{\theta} = Vl\theta - Hl$ (3–13) $\quad m(\ddot{x} + l\ddot{\theta}) = H$ (3–14) $\quad 0 = V - mg$ (3–15)

From Equations (3–12) and (3–14), we obtain $\quad (M + m)\ddot{x} + ml\ddot{\theta} = u$ (3–16)

From Equations (3–13), (3–14), and (3–15), we have

$I\ddot{\theta} = mgl\theta - Hl = mgl\theta - l(m\ddot{x} + ml\ddot{\theta})$ or $(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta$ (3–17)

Equations (3–16) and (3–17) describe the motion of the inverted-pendulum-on-the-cart system. They constitute a mathematical model of the system.

EXAMPLE 3–6

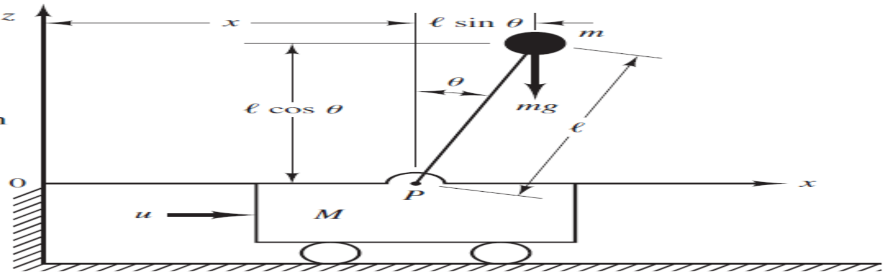
Consider the inverted-pendulum system shown in Figure 3–6. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$ in Equation (3–17). Then the mathematical model for this system becomes as follows:

$(M + m)\ddot{x} + ml\ddot{\theta} = u$ (3–18) $\quad ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta$ (3–19)

Equations (3–18) and (3–19) can be modified to

$Ml\ddot{\theta} = (M + m)g\theta - u$ (3–20) $\quad M\ddot{x} = u - mg\theta$ (3–21)

Figure 3–6 Inverted-pendulum system.



Equation (3–20) was obtained by eliminating $\ddot{x}$ from Equations (3–18) and (3–19). Equation (3–21) was obtained by eliminating $\ddot{\theta}$ from Equations (3–18) and (3–19). From Equation (3–20) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M + m)g} = \frac{1}{Ml\left(s + \sqrt{\frac{M + m}{Ml}g}\right)\left(s - \sqrt{\frac{M + m}{Ml}g}\right)}$$

The inverted-pendulum plant has one pole on the negative real axis $[s = -(\sqrt{M + m}/\sqrt{Ml})\sqrt{g}]$ and another on the positive real axis $[s = (\sqrt{M + m}/\sqrt{Ml})\sqrt{g}]$. Hence, the plant is open-loop unstable.

Define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta \qquad x_2 = \dot{\theta} \qquad x_3 = x \qquad x_4 = \dot{x}$$

Note that angle θ indicates the rotation of the pendulum rod about point P, and x is the location of the cart. If we consider θ and x as the outputs of the system, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

(Notice that both θ and x are easily measurable quantities.) Then, from the definition of the state variables and Equations (3–20) and (3–21), we obtain

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \frac{M + m}{Ml}gx_1 - \frac{1}{Ml}u \qquad \dot{x}_3 = x_4 \qquad \dot{x}_4 = -\frac{m}{M}gx_1 + \frac{1}{M}u$$

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M + m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} u \tag{3–22}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3–23}$$

Equations (3–22) and (3–23) give a state-space representation of the inverted-pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations for this system.)

3–3 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node.) Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. (This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop.) A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

This section first deals with simple electrical circuits and then treats mathematical modeling of operational amplifier systems.

***LRC* Circuit.** Consider the electrical circuit shown in Figure 3–7. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \qquad (3\text{–}24)$$

$$\frac{1}{C}\int i\,dt = e_o \qquad (3\text{–}25)$$

Figure 3–7 Electrical circuit.

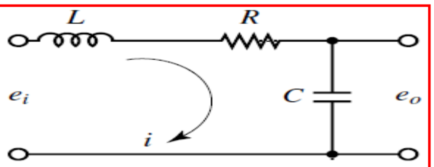
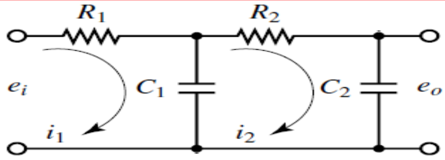


Figure 3–8 Electrical system.



Equations (3–24) and (3–25) give a mathematical model of the circuit.

A transfer-function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (3–24) and (3–25), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s) \qquad \frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \qquad (3\text{–}26)$$

A state-space model of the system shown in Figure 3–7 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (3–26) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by $x_1 = e_o$ $\quad x_2 = \dot{e}_o$

and the input and output variables by $u = e_i$ $\quad y = e_o = x_1$

we obtain
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}u \quad \text{and} \quad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascaded Elements. Many feedback systems have components that load each other. Consider the system shown in Figure 3–8. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially.

It will be shown that the second stage of the circuit (R_2C_2 portion) produces a loading effect on the first stage (R_1C_1 portion). The equations for this system are

$$\frac{1}{C_1}\int(i_1 - i_2)\,dt + R_1i_1 = e_i \qquad (3\text{–}27)$$

and
$$\frac{1}{C_1}\int(i_2 - i_1)\,dt + R_2i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \qquad (3\text{–}28) \qquad \frac{1}{C_2}\int i_2\,dt = e_o \qquad (3\text{–}29)$$

Taking the Laplace transforms of Equations (3–27) through (3–29), respectively, using zero initial conditions, we obtain

$$\frac{1}{C_1s}[I_1(s) - I_2(s)] + R_1I_1(s) = E_i(s) \qquad (3\text{–}30)$$

$$\frac{1}{C_1s}[I_2(s) - I_1(s)] + R_2I_2(s) + \frac{1}{C_2s}I_2(s) = 0 \qquad (3\text{–}31) \qquad \frac{1}{C_2s}I_2(s) = E_o(s) \qquad (3\text{–}32)$$

Eliminating $I_1(s)$ from Equations (3–30) and (3–31) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s}$$
$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1} \qquad (3\text{–}33)$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since $(R_1C_1 + R_2C_2 + R_1C_2)^2 > 4R_1C_1R_2C_2$, the two roots of the denominator of Equation (3–33) are real.

The present analysis shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s + 1)$ and $1/(R_2C_2s + 1)$. The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

Complex Impedances. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations. Consider the system shown in Figure 3–9(a). In this system, Z_1 and Z_2 represent complex impedances. The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals, to $I(s)$, the Laplace transform of the current through the element, under the assumption that the initial conditions are zero, so that $Z(s) = E(s)/I(s)$. If the two-terminal element is a resistance R, capacitance C, or inductance L, then the complex impedance is given by R, $1/Cs$, or Ls, respectively. If complex impedances are connected in series, the total impedance is the sum of the individual complex impedances.

Remember that the impedance approach is valid only if the initial conditions involved are all zeros. Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit. This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Consider the circuit shown in Figure 3–9(b). Assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the transfer function of this circuit is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

For the system shown in Figure 3–7, $\quad Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$

Hence the transfer function $E_o(s)/E_i(s)$ can be found as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

which is, of course, identical to Equation (3–26).

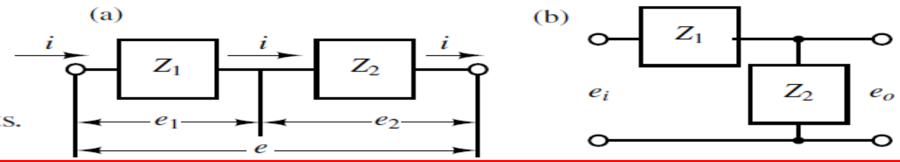


Figure 3–9 Electrical circuits.

EXAMPLE 3–7

Consider again the system shown in Figure 3–8. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the complex impedance approach. (Capacitors C_1 and C_2 are not charged initially.)

The circuit shown in Figure 3–8 can be redrawn as that shown in Figure 3–10(a), which can be further modified to Figure 3–10(b).

In the system shown in Figure 3–10(b) the current I is divided into two currents I_1 and I_2. Noting that

$$Z_2 I_1 = (Z_3 + Z_4)I_2, \qquad I_1 + I_2 = I$$

we obtain

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

Noting that

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C_1 s}\frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$

which is the same as that given by Equation (3–33).

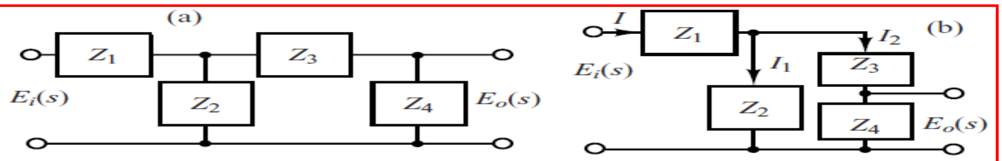


Figure 3–10 (a) The circuit of Figure 3–8 shown in terms of impedances; (b) equivalent circuit diagram.

Transfer Functions of Nonloading Cascaded Elements. The transfer function of a system consisting of two nonloading cascaded elements can be obtained by eliminating the intermediate input and output. For example, consider the system shown in Figure 3–11(a). The transfer functions of the elements are

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \qquad \text{and} \qquad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element. Then the transfer function of the whole system becomes

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)} = G_1(s)G_2(s)$$

The transfer function of the whole system is thus the product of the transfer functions of the individual elements. This is shown in Figure 3–11(b).

As an example, consider the system shown in Figure 3–12. The insertion of an isolating amplifier between the circuits to obtain nonloading characteristics is frequently used in combining circuits. Since amplifiers have very high input impedances, an isolation amplifier inserted between the two circuits justifies the nonloading assumption.

The two simple RC circuits, isolated by an amplifier as shown in Figure 3–12, have negligible loading effects, and the transfer function of the entire circuit equals the product of the individual transfer functions. Thus, in this case,

$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{R_1C_1s + 1}\right)(K)\left(\frac{1}{R_2C_2s + 1}\right) = \frac{K}{(R_1C_1s + 1)(R_2C_2s + 1)}$$

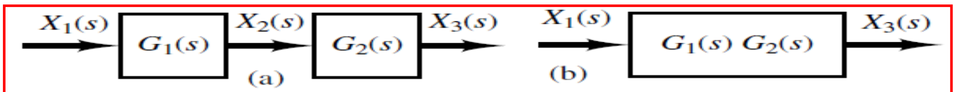


Figure 3–11 (a) System consisting of two nonloading cascaded elements; (b) an equivalent system.

Electronic Controllers. In what follows we shall discuss electronic controllers using operational amplifiers. We begin by deriving the transfer functions of simple operational-amplifier circuits. Then we derive the transfer functions of some of the operational-amplifier controllers. Finally, we give operational-amplifier controllers and their transfer functions in the form of a table.

Figure 3–12 Electrical system.

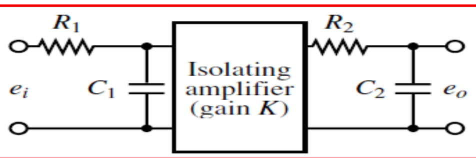
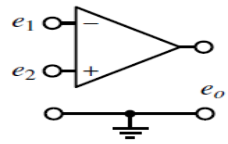


Figure 3–13 Operational amplifier.



Operational Amplifiers. Operational amplifiers, often called op amps, are frequently used to amplify signals in sensor circuits. Op amps are also frequently used in filters used for compensation purposes. Figure 3–13 shows an op amp. It is a common practice to choose the ground as 0 volt and measure the input voltages e_1 and e_2 relative to the ground. The input e_1 to the minus terminal of the amplifier is inverted, and the input e_2 to the plus terminal is not inverted. The total input to the amplifier thus becomes $e_2 - e_1$. Hence, for the circuit shown in Figure 3–13, we have

$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

where the inputs e_1 and e_2 may be dc or ac signals and K is the differential gain (voltage gain). The magnitude of K is approximately $10^5 \sim 10^6$ for dc signals and ac signals with frequencies less than approximately 10 Hz. (The differential gain K decreases with the signal frequency and becomes about unity for frequencies of 1 MHz ~ 50 MHz.) Note that the op amp amplifies the difference in voltages e_1 and e_2. Such an amplifier is commonly called a differential amplifier. Since the gain of the op amp is very high, it is necessary to have a negative feedback from the output to the input to make the amplifier stable. (The feedback is made from the output to the inverted input so that the feedback is a negative feedback.)

In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In an actual op amp, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, we make the assumption that the op amps are ideal.

Inverting Amplifier. Consider the operational-amplifier circuit shown in Figure 3–14. Let us obtain the output voltage e_o.

The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

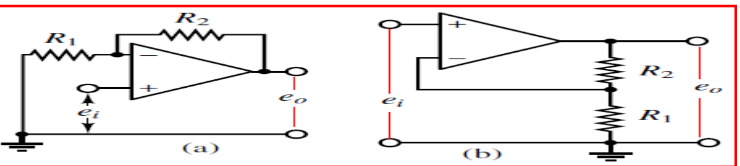
$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero, or $e' \doteq 0$. Hence we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2} \qquad \text{or} \qquad e_o = -\frac{R_2}{R_1}e_i$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.

Figure 3–15 (a) Noninverting operational amplifier; (b) equivalent circuit.



EXAMPLE 3–8

Figure 3–16 shows an electrical circuit involving an operational amplifier. Obtain the output e_o.

Let us define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d(e' - e_o)}{dt}, \qquad i_3 = \frac{e' - e_o}{R_2}$$

Noting that the current flowing into the amplifier is negligible, we have $i_1 = i_2 + i_3$

Hence $\dfrac{e_i - e'}{R_1} = C\dfrac{d(e' - e_o)}{dt} + \dfrac{e' - e_o}{R_2}$ Since $e' \doteq 0$, we have $\dfrac{e_i}{R_1} = -C\dfrac{de_o}{dt} - \dfrac{e_o}{R_2}$

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have

$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s) \qquad \text{which can be written as} \qquad \frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

The op-amp circuit shown in Figure 3–16 is a first-order lag circuit. (Several other circuits involving op amps are shown in Table 3–1 together with their transfer functions. Table 3–1 is given on page 85.)

Figure 3–16 First-order lag circuit using operational amplifier.

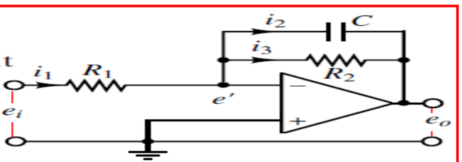
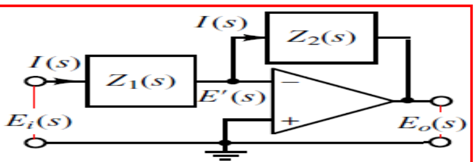


Figure 3–17 Operational-amplifier circuit.



Impedance Approach to Obtaining Transfer Functions. Consider the op-amp circuit shown in Figure 3–17. Similar to the case of electrical circuits we discussed earlier, the impedance approach can be applied to op-amp circuits to obtain their transfer functions. For the circuit shown in Figure 3–17, we have

$$\frac{E_i(s) - E'(s)}{Z_1} = \frac{E'(s) - E_o(s)}{Z_2} \quad \text{Since } E'(s) \doteq 0, \text{ we have} \quad \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \qquad (3\text{–}34)$$

EXAMPLE 3–9

Referring to the op-amp circuit shown in Figure 3–16, obtain the transfer function $E_o(s)/E_i(s)$ by use of the impedance approach. The complex impedances $Z_1(s)$ and $Z_2(s)$ for this circuit are

$$Z_1(s) = R_1 \qquad \text{and} \qquad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

The transfer function $E_o(s)/E_i(s)$ is, therefore, obtained as

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

where $\qquad T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \dfrac{R_4C_1}{R_3C_2}$

Notice that $\qquad K_c\alpha = \dfrac{R_4C_1}{R_3C_2}\dfrac{R_2C_2}{R_1C_1} = \dfrac{R_2R_4}{R_1R_3}, \qquad \alpha = \dfrac{R_2C_2}{R_1C_1}$

This network has a dc gain of $K_c\alpha = R_2R_4/(R_1R_3)$.

Note that this network, whose transfer function is given by Equation (3–36), is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$.

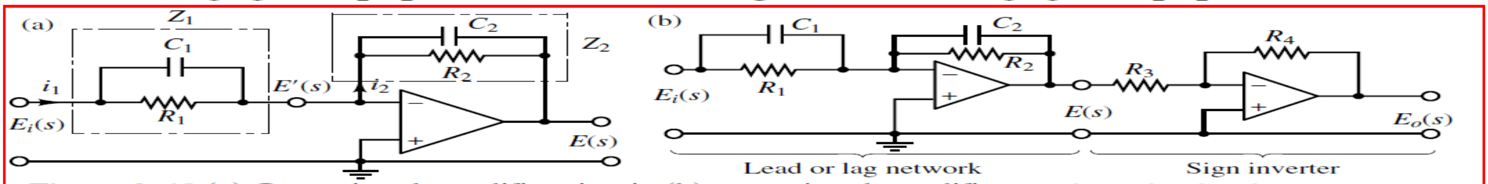


Figure 3–18 (a) Operational-amplifier circuit; (b) operational-amplifier used as a lead or lag compensator.

PID Controller Using Operational Amplifiers. Figure 3–19 shows an electronic proportional-plus-integral-plus-derivative controller (a PID controller) using operational amplifiers. The transfer function $E(s)/E_i(s)$ is given by

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1C_1s + 1}, \qquad Z_2 = \frac{R_2C_2s + 1}{C_2s}$$

Thus

$$\frac{E(s)}{E_i(s)} = -\left(\frac{R_2C_2s + 1}{C_2s}\right)\left(\frac{R_1C_1s + 1}{R_1}\right)$$

Noting that

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

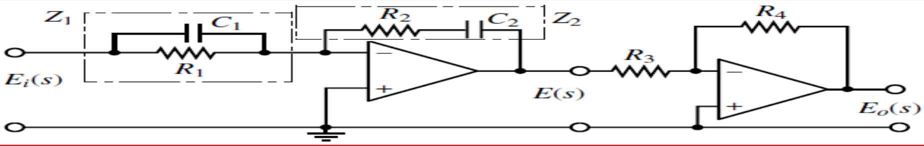


Figure 3–19 Electronic PID controller.

we have

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} = \frac{R_4R_2(R_1C_1s+1)(R_2C_2s+1)}{R_3R_1 \quad R_2C_2s} = \frac{R_4R_2}{R_3R_1}\left(\frac{R_1C_1+R_2C_2}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1s\right)$$

$$= \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}\left[1 + \frac{1}{(R_1C_1 + R_2C_2)s} + \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}s\right] \qquad (3\text{–}37)$$

Notice that the second operational-amplifier circuit acts as a sign inverter as well as a gain adjuster. When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p\left(1 + \frac{T_i}{s} + T_ds\right)$$

K_p is called the proportional gain, T_i is called the integral time, and T_d is called the derivative time. From Equation (3–37) we obtain the proportional gain K_p, integral time T_i, and derivative time T_d to be

$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2} \qquad T_d = \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2} \qquad T_i = \frac{1}{R_1C_1 + R_2C_2}$$

When a PID controller is expressed as

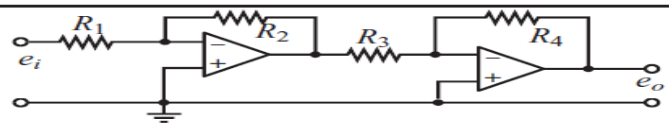
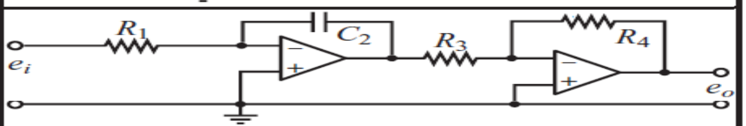
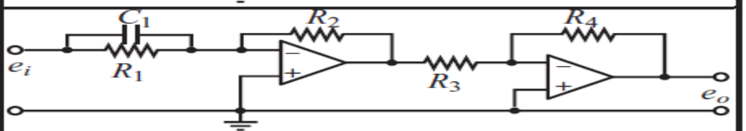
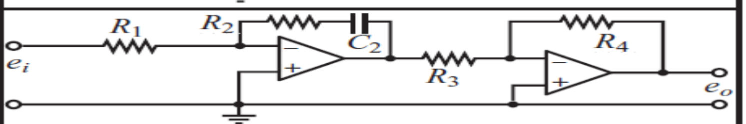
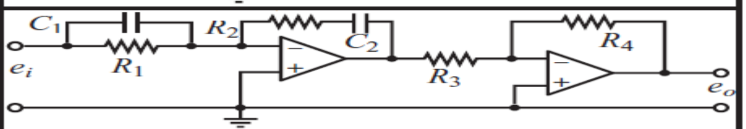
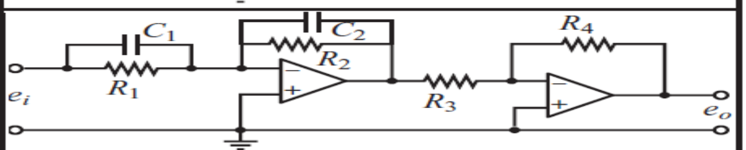
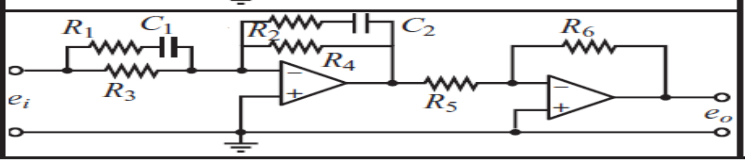
$$\boxed{\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_ds}$$

K_p is called the proportional gain, K_i is called the integral gain, and K_d is called the derivative gain. For this controller

$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2} \qquad K_i = \frac{R_4}{R_3R_1C_2} \qquad K_d = \frac{R_4R_2C_1}{R_3}$$

Table 3–1 shows a list of operational-amplifier circuits that may be used as controllers or compensators.

Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s + 1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s + 1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s + 1}{R_2C_2s + 1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1 + R_3)C_1s + 1](R_2C_2s + 1)}{(R_1C_1s + 1)[(R_2 + R_4)C_2s + 1]}$	

EXAMPLE PROBLEMS AND SOLUTIONS

A–3–1.
Figure 3–20(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Figure 3–20(b). Assuming that the motion x_i at point P is the input to the system and the vertical motion x_o of the body is the output, obtain the transfer function $X_o(s)/X_i(s)$. (Consider the motion of the body only in the vertical direction.) Displacement x_o is measured from the equilibrium position in the absence of input x_i.

Solution. The equation of motion for the system shown in Figure 3–20(b) is

$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$ or $m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

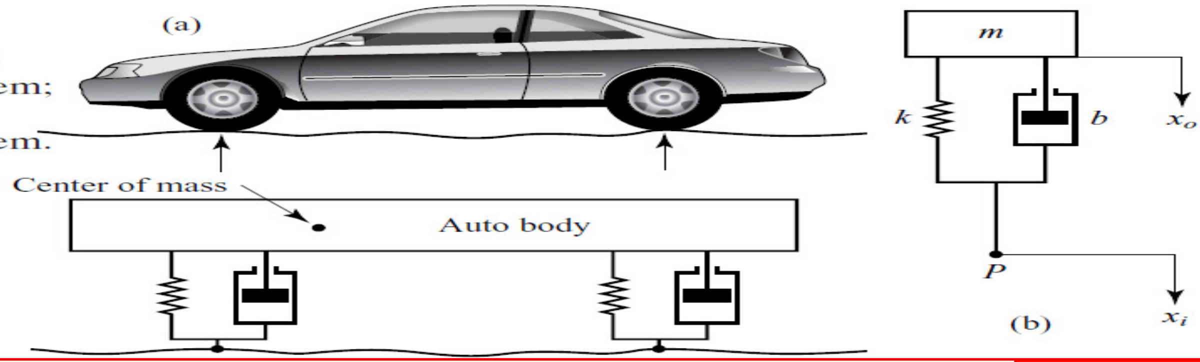


Figure 3–20
(a) Automobile suspension system; (b) simplified suspension system.

A–3–2.
Obtain the transfer function $Y(s)/U(s)$ of the system shown in Figure 3–21. The input u is a displacement input. (Like the system of Problem **A–3–1**, this is also a simplified version of an automobile or motorcycle suspension system.)

Solution. Assume that displacements x and y are measured from respective steady-state positions in the absence of input u. Applying Newton's second law to this system, we obtain

$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$ $\quad m_2\ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$

Hence, we have $m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2y + k_1u$ $\quad m_2\ddot{y} + b\dot{y} + k_2y = b\dot{x} + k_2x$

Taking Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$[m_1s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1U(s)$ $\quad [m_2s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1s^2 + bs + k_1 + k_2)\frac{m_2s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1U(s)$$ which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1m_2s^4 + (m_1 + m_2)bs^3 + [k_1m_2 + (m_1 + m_2)k_2]s^2 + k_1bs + k_1k_2}$$

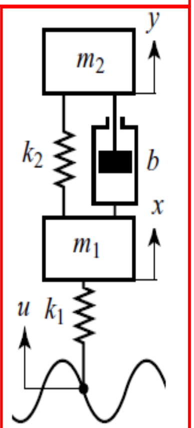


Figure 3–21
Suspension system

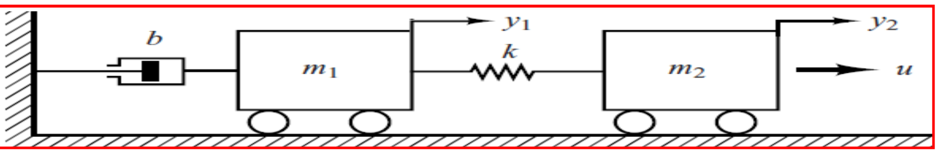


Figure 3–22
Mechanical system.

A–3–3.
Obtain a state-space representation of the system shown in Figure 3–22.

Solution. The system equations are

$m_1\ddot{y}_1 + b\dot{y}_1 + k(y_1 - y_2) = 0$ $\quad m_2\ddot{y}_2 + k(y_2 - y_1) = u$

The output variables for this system are y_1 and y_2. Define state variables as

$x_1 = y_1$ $\quad x_2 = \dot{y}_1$ $\quad x_3 = y_2$ $\quad x_4 = \dot{y}_2$

Then we obtain the following equations:

$\dot{x}_1 = x_2$ $\quad \dot{x}_2 = \frac{1}{m_1}[-b\dot{y}_1 - k(y_1 - y_2)] = -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3$

$\dot{x}_3 = x_4$ $\quad \dot{x}_4 = \frac{1}{m_2}[-k(y_2 - y_1) + u] = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}u$

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$

and the output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A–3–4.
Obtain the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 3–23(a). Also obtain the transfer function $E_o(s)/E_i(s)$ of the electrical system shown in Figure 3–23(b). Show that these transfer functions of the two systems are of identical form and thus they are analogous systems.

Solution. In Figure 3–23(a) we assume that displacements x_i, x_o, and y are measured from their respective steady-state positions. Then the equations of motion for the mechanical system shown in Figure 3–23(a) are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y}) \qquad b_2(\dot{x}_o - \dot{y}) = k_2 y$$

By taking the Laplace transforms of these two equations, assuming zero initial conditions, we have

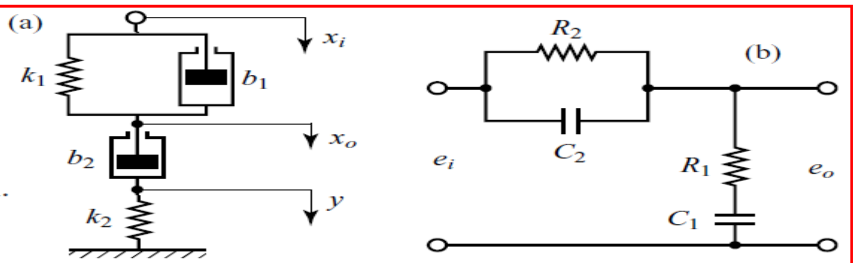
$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)] \qquad b_2[sX_o(s) - sY(s)] = k_2Y(s)$$

If we eliminate $Y(s)$ from the last two equations, then we obtain

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2sX_o(s) - b_2s\frac{b_2sX_o(s)}{b_2s + k_2}$$

or $$(b_1s + k_1)X_i(s) = \left(b_1s + k_1 + b_2s - b_2s\frac{b_2s}{b_2s + k_2}\right)X_o(s)$$

Figure 3–23 (a) Mechanical system; (b) analogous electrical system.



Hence the transfer function $X_o(s)/X_i(s)$ can be obtained as

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

For the electrical system shown in Figure 3–23(b), the transfer function $E_o(s)/E_i(s)$ is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 + 1/C_1s}{1/(1/R_2) + C_2s \quad + \quad R_1 + 1/C_1s} = \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{(R_1C_1s + 1)(R_2C_2s + 1) + R_2C_1s}$$

A comparison of the transfer functions shows that the systems shown in Figures 3–23(a) and (b) are analogous.

A–3–5.
Obtain transfer functions $E_o(s)/E_i(s)$ of bridged T networks shown in Figures 3–24(a) and (b).

Solution. The bridged T networks shown can both be represented by the network of Figure 3–25(a), where we used complex impedances. This network may be modified to that shown in Figure 3–25(b). In Figure 3–25(b), note that $I_1 = I_2 + I_3$, $\quad I_2Z_1 = (Z_3 + Z_4)I_3$

Figure 3–24 Bridged T networks.

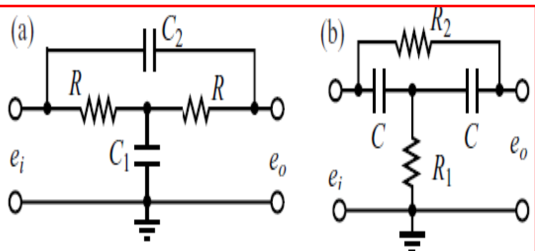
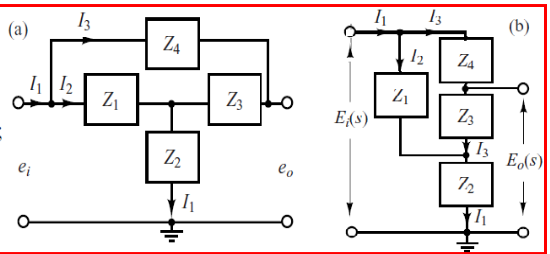


Figure 3–25 (a) Bridged T network in terms of complex impedances; (b) equivalent network.



Hence $$I_2 = \frac{Z_3 + Z_4}{Z_1 + Z_3 + Z_4}I_1, \qquad I_3 = \frac{Z_1}{Z_1 + Z_3 + Z_4}I_1$$

Then the voltages $E_i(s)$ and $E_o(s)$ can be obtained as

$$E_i(s) = Z_1I_2 + Z_2I_1 = \left[Z_2 + \frac{Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4}\right]I_1 = \frac{Z_2(Z_1 + Z_3 + Z_4) + Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4}I_1$$

$$E_o(s) = Z_3I_3 + Z_2I_1 = \frac{Z_3Z_1}{Z_1 + Z_3 + Z_4}I_1 + Z_2I_1 = \frac{Z_3Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_1 + Z_3 + Z_4}I_1$$

Hence, the transfer function $E_o(s)/E_i(s)$ of the network shown in Figure 3–25(a) is obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_3Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_2(Z_1 + Z_3 + Z_4) + Z_1Z_3 + Z_1Z_4} \qquad (3\text{–}38)$$

For the bridged T network shown in Figure 3–24(a), substitute

$$Z_1 = R, \qquad Z_2 = \frac{1}{C_1s}, \qquad Z_3 = R, \qquad Z_4 = \frac{1}{C_2s}$$

into Equation (3–38). Then we obtain the transfer function $E_o(s)/E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = \frac{R^2 + 1/C_1s\,[R + R + 1/C_2s]}{1/C_1s\,[R + R + 1/C_2s] + R^2 + R1/C_2s} = \frac{RC_1RC_2s^2 + 2RC_2s + 1}{RC_1RC_2s^2 + (2RC_2 + RC_1)s + 1}$$

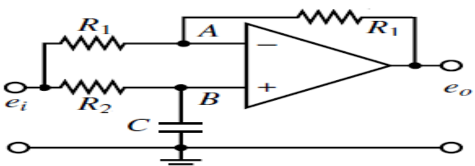
Similarly, for the bridged T network shown in Figure 3–24(b), we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_1, \qquad Z_3 = \frac{1}{Cs}, \qquad Z_4 = R_2$$

into Equation (3–38). Then the transfer function $E_o(s)/E_i(s)$ can be obtained as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}\frac{1}{Cs} + R_1\left(\frac{1}{Cs} + \frac{1}{Cs} + R_2\right)}{R_1\left(\frac{1}{Cs} + \frac{1}{Cs} + R_2\right) + \frac{1}{Cs}\frac{1}{Cs} + R_2\frac{1}{Cs}} = \frac{R_1CR_2Cs^2 + 2R_1Cs + 1}{R_1CR_2Cs^2 + (2R_1C + R_2C)s + 1}$$

Figure 3–26 Operational-amplifier circuit.



A–3–6.
Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3–26.

Solution. The voltage at point A is $e_A = \frac{1}{2}(e_i - e_o) + e_o$

The Laplace-transformed version of this last equation is $E_A(s) = \frac{1}{2}[E_i(s) + E_o(s)]$

The voltage at point B is $E_B(s) = \dfrac{1/Cs}{R_2 + 1/Cs} E_i(s) = \dfrac{1}{R_2Cs + 1} E_i(s)$

Since $[E_B(s) - E_A(s)]K = E_o(s)$ and $K \gg 1$, we must have $E_A(s) = E_B(s)$. Thus

Hence $\frac{1}{2}[E_i(s) + E_o(s)] = \dfrac{1}{R_2Cs + 1} E_i(s)$ $\quad \dfrac{E_o(s)}{E_i(s)} = -\dfrac{R_2Cs - 1}{R_2Cs + 1} = -\dfrac{s - 1/R_2C}{s + 1/R_2C}$

A–3–7.
Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp system shown in Figure 3–27 in terms of complex impedances Z_1, Z_2, Z_3, and Z_4. Using the equation derived, obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp system shown in Figure 3–26.

Solution. From Figure 3–27, we find $\dfrac{E_i(s) - E_A(s)}{Z_3} = \dfrac{E_A(s) - E_o(s)}{Z_4}$

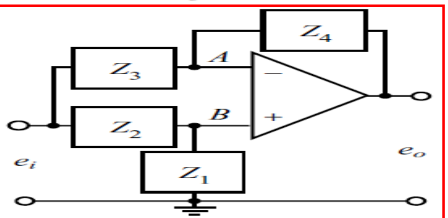


Figure 3–27 Operational-amplifier circuit.

or $E_i(s) - \left(1 + \dfrac{Z_3}{Z_4}\right) E_A(s) = -\dfrac{Z_3}{Z_4} E_o(s)$ $\quad$ Since $E_A(s) = E_B(s) = \dfrac{Z_1}{Z_1 + Z_2} E_i(s)$ $\quad$ (3–39)

by substituting Equation (3–40) into Equation (3–39), we obtain

$$\left[\frac{Z_4Z_1 + Z_4Z_2 - Z_4Z_1 - Z_3Z_1}{Z_4(Z_1 + Z_2)}\right] E_i(s) = -\frac{Z_3}{Z_4} E_o(s) \qquad (3\text{–}40)$$

from which we get the transfer function $E_o(s)/E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_4Z_2 - Z_3Z_1}{Z_3(Z_1 + Z_2)} \qquad (3\text{–}41)$$

To find the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 3–26, we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_2, \qquad Z_3 = R_1, \qquad Z_4 = R_1$$

into Equation (3–41). The result is $\dfrac{E_o(s)}{E_i(s)} = -\dfrac{R_1R_2 - R_1 1/Cs}{R_1(1/Cs + R_2)} = -\dfrac{R_2Cs - 1}{R_2Cs + 1}$

which is, as a matter of course, the same as that obtained in Problem **A–3–6.**

A–3–8.
Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 3–28.

Solution. We will first obtain currents i_1, i_2, i_3, i_4, and i_5. Then we will use node equations at nodes A and B.

$$i_1 = \frac{e_i - e_A}{R_1}; \qquad i_2 = \frac{e_A - e_o}{R_3}, \qquad i_3 = C_1\frac{de_A}{dt} \qquad i_4 = \frac{e_A}{R_2}, \qquad i_5 = C_2\frac{-de_o}{dt}$$

At node A, we have $i_1 = i_2 + i_3 + i_4$, or $\quad \dfrac{e_i - e_A}{R_1} = \dfrac{e_A - e_o}{R_3} + C_1\dfrac{de_A}{dt} + \dfrac{e_A}{R_2}$ $\quad$ (3–42)

At node B, we get $i_4 = i_5$, or $\quad \dfrac{e_A}{R_2} = C_2\dfrac{-de_o}{dt}$ $\quad$ (3–43)

By rewriting Equation (3–42), we have $\quad C_1\dfrac{de_A}{dt} + \left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right)e_A = \dfrac{e_i}{R_1} + \dfrac{e_o}{R_3}$ $\quad$ (3–44)

From Equation (3–43), we get $\quad e_A = -R_2C_2\dfrac{de_o}{dt}$ $\quad$ (3–45)

By substituting Equation (3–45) into Equation (3–44), we obtain

$$C_1\left(-R_2C_2\frac{d^2e_o}{dt^2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)\frac{de_o}{dt} = \frac{e_i}{R_1} + \frac{e_o}{R_3}$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$-C_1C_2R_2s^2E_o(s) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)sE_o(s) - \frac{1}{R_3}E_o(s) = \frac{E_i(s)}{R_1}$$

from which we get the transfer function $E_o(s)/E_i(s)$ as follows:

$$\frac{E_o(s)}{E_i(s)} = -\frac{1}{R_1C_1R_2C_2s^2 + [R_2C_2 + R_1C_2 + (R_1/R_3)R_2C_2]s + (R_1/R_3)}$$

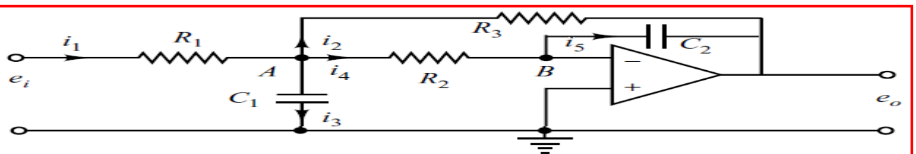


Figure 3–28 Operational-amplifier circuit.

A–3–9.
Consider the servo system shown in Figure 3–29(a). The motor shown is a servomotor, a dc motor designed specifically to be used in a control system. The operation of this system is as follows: A pair of potentiometers acts as an error-measuring device. They convert the input and output positions into proportional electric signals. The command input signal determines the angular position r of the wiper arm of the input potentiometer. The angular position r is the reference input to the system, and the electric potential of the arm is proportional to the angular position of the arm. The output shaft position determines the angular position c of the wiper arm of the output potentiometer. The difference between the input angular position r and the output angular position c is the error signal e, or

$$e = r - c$$

The potential difference $e_r - e_c = e_v$ is the error voltage, where e_r is proportional to r and e_c is proportional to c; that is, $e_r = K_0 r$ and $e_c = K_0 c$, where K_0 is a proportionality constant. The error voltage that appears at the potentiometer terminals is amplified by the amplifier whose gain constant is K_1. The output voltage of this amplifier is applied to the armature circuit of the dc motor. A fixed voltage is applied to the field winding. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero. For constant field current, the torque developed by the motor is

$$T = K_2 i_a$$

where K_2 is the motor torque constant and i_a is the armature current.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage e_b is directly proportional to the angular velocity $d\theta/dt$, or

$$e_b = K_3 \frac{d\theta}{dt}$$

where e_b is the back emf, K_3 is the back emf constant of the motor, and θ is the angular displacement of the motor shaft.

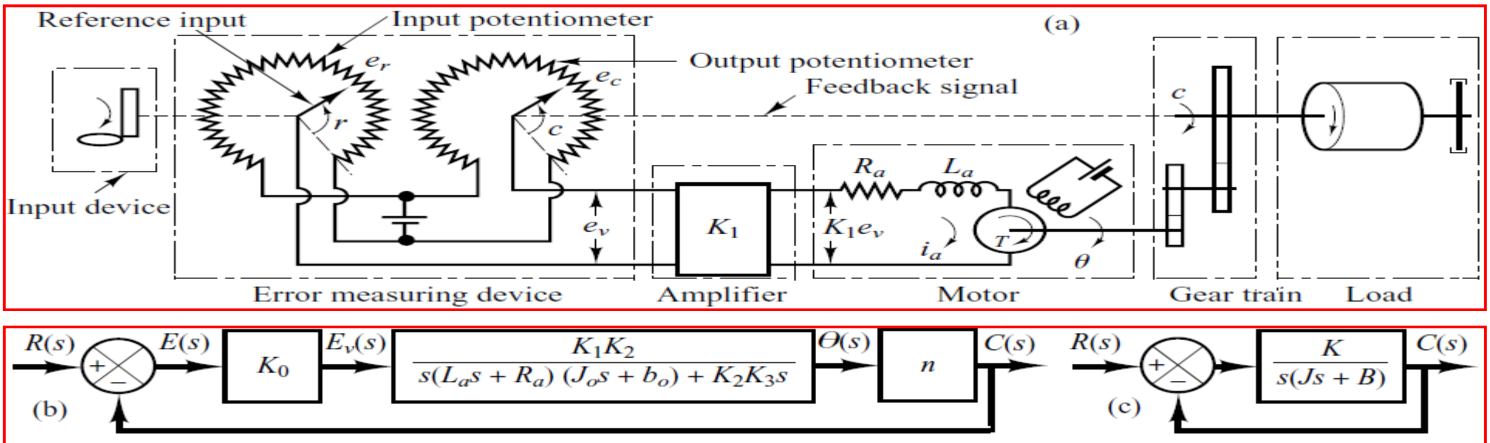


Figure 3–29 (a) Schematic of servo system; (b) block diagram ; (c) simplified block diagram.

Obtain the transfer function between the motor shaft angular displacement θ and the error voltage e_v. Obtain also a block diagram for this system and a simplified block diagram when L_a is negligible.

Solution. The speed of an armature-controlled dc servomotor is controlled by the armature voltage e_a. (The armature voltage $e_a = K_1 e_v$ is the output of the amplifier.) The differential equation for the armature circuit is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \quad \text{or} \quad L_a \frac{di_a}{dt} + R_a i_a + K_3 \frac{d\theta}{dt} = K_1 e_v \tag{3–46}$$

The equation for torque equilibrium is

$$J_0 \frac{d^2\theta}{dt^2} + b_0 \frac{d\theta}{dt} = T = K_2 i_a \tag{3–47}$$

where J_0 is the inertia of the combination of the motor, load, and gear train referred to the motor shaft and b_0 is the viscous-friction coefficient of the combination of the motor, load, and gear train referred to the motor shaft.

By eliminating i_a from Equations (3–46) and (3–47), we obtain

$$\frac{\Theta(s)}{E_v(s)} = \frac{K_1 K_2}{s(L_a s + R_a)(J_0 s + b_0) + K_2 K_3 s} \tag{3–48}$$

We assume that the gear ratio of the gear train is such that the output shaft rotates n times for each revolution of the motor shaft. Thus,

$$C(s) = n\Theta(s) \tag{3–49}$$

The relationship among $E_v(s)$, $R(s)$, and $C(s)$ is

$$E_v(s) = K_0[R(s) - C(s)] = K_0 E(s) \tag{3–50}$$

The block diagram of this system can be constructed from Equations (3–48), (3–49), and (3–50), as shown in Figure 3–29(b). The transfer function in the feedforward path of this system is

$$G(s) = \frac{C(s)}{\Theta(s)} \frac{\Theta(s)}{E_v(s)} \frac{E_v(s)}{E(s)} = \frac{K_0 K_1 K_2 n}{s[(L_a s + R_a)(J_0 s + b_0) + K_2 K_3]}$$

When L_a is small, it can be neglected, and the transfer function $G(s)$ in the feedforward path becomes

$$G(s) = \frac{K_0 K_1 K_2 n}{s[R_a(J_0 s + b_0) + K_2 K_3]} = \frac{K_0 K_1 K_2 n / R_a}{J_0 s^2 + \left(b_0 + K_2 K_3 / R_a\right) s} \tag{3–51}$$

The term $[b_0 + (K_2K_3/R_a)]s$ indicates that the back emf of the motor effectively increases the viscous friction of the system. The inertia J_0 and viscous friction coefficient $b_0 + (K_2K_3/R_a)$ are referred to the motor shaft. When J_0 and $b_0 + (K_2K_3/R_a)$ are multiplied by $1/n^2$, the inertia and viscous-friction coefficient are expressed in terms of the output shaft. Introducing new parameters defined by

$J = J_0/n^2$ = moment of inertia referred to the output shaft $\qquad K = K_0 K_1 K_2 / n R_a$

$B = [b_0 + (K_2K_3/R_a)]/n^2$ = viscous-friction coefficient referred to the output shaft

the transfer function $G(s)$ given by Equation (3–51) can be simplified, yielding

$$G(s) = \frac{K}{Js^2 + Bs} \quad \text{or} \quad G(s) = \frac{K_m}{s(T_m s + 1)} \qquad \text{where } K_m = \frac{K}{B},\ T_m = \frac{J}{B} = \frac{R_a J_0}{R_a b_0 + K_2 K_3}$$

The block diagram of the system shown in Figure 3–29(b) can thus be simplified as shown in Figure 3–29(c).

****

Mathematical Modeling of Fluid and Thermal Systems 4

4–1 INTRODUCTION

This chapter treats mathematical modeling of fluid systems and thermal systems. As the most versatile medium for transmitting signals and power, fluids—liquids and gases—have wide usage in industry. Liquids and gases can be distinguished basically by their relative incompressibilities and the fact that a liquid may have a free surface, whereas a gas expands to fill its vessel. In the engineering field the term *pneumatic* describes fluid systems that use air or gases and *hydraulic* applies to those using oil.

We first discuss liquid-level systems that are frequently used in process control. Here we introduce the concepts of resistance and capacitance to describe the dynamics of such systems. Then we treat pneumatic systems. Such systems are extensively used in the automation of production machinery and in the field of automatic controllers. For instance, pneumatic circuits that convert the energy of compressed air into mechanical energy enjoy wide usage. Also, various types of pneumatic controllers are widely used in industry. Next, we present hydraulic servo systems. These are widely used in machine tool systems, aircraft control systems, etc. We discuss basic aspects of hydraulic servo systems and hydraulic controllers. Both pneumatic systems and hydraulic systems can be modeled easily by using the concepts of resistance and capacitance. Finally, we treat simple thermal systems. Such systems involve heat transfer from one substance to another. Mathematical models of such systems can be obtained by using thermal resistance and thermal capacitance.

Outline of the Chapter. Section 4–1 has presented introductory material for the chapter. Section 4–2 discusses liquid-level systems. Section 4–3 treats pneumatic systems—in particular, the basic principles of pneumatic controllers. Section 4–4 first discusses hydraulic servo systems and then presents hydraulic controllers. Finally, Section 4–5 analyzes thermal systems and obtains mathematical models of such systems.

4–2 LIQUID-LEVEL SYSTEMS

In analyzing systems involving fluid flow, we find it necessary to divide flow regimes into laminar flow and turbulent flow, according to the magnitude of the Reynolds number. If the Reynolds number is greater than about 3000 to 4000, then the flow is turbulent. The flow is laminar if the Reynolds number is less than about 2000. In the laminar case, fluid flow occurs in streamlines with no turbulence. Systems involving laminar flow may be represented by linear differential equations.

Industrial processes often involve flow of liquids through connecting pipes and tanks. The flow in such processes is often turbulent and not laminar. Systems involving turbulent flow often have to be represented by nonlinear differential equations. If the region of operation is limited, however, such nonlinear differential equations can be linearized. We shall discuss such linearized mathematical models of liquid-level systems in this section. Note that the introduction of concepts of resistance and capacitance for such liquid-level systems enables us to describe their dynamic characteristics in simple forms.

Resistance and Capacitance of Liquid-Level Systems. Consider the flow through a short pipe connecting two tanks. The resistance R for liquid flow in such a pipe or restriction is defined as the change in the level difference (the difference of the liquid levels of the two tanks) necessary to cause a unit change in flow rate; that is,

$$R = \frac{\text{change in level difference, m}}{\text{change in flow rate, m}^3/\text{sec}}$$

Since the relationship between the flow rate and level difference differs for the laminar flow and turbulent flow, we shall consider both cases in the following.

Consider the liquid-level system shown in Figure 4–1(a). In this system the liquid spouts through the load valve in the side of the tank. If the flow through this restriction is laminar, the relationship between the steady-state flow rate and steady-state head at the level of the restriction is given by $\boxed{Q = KH}$

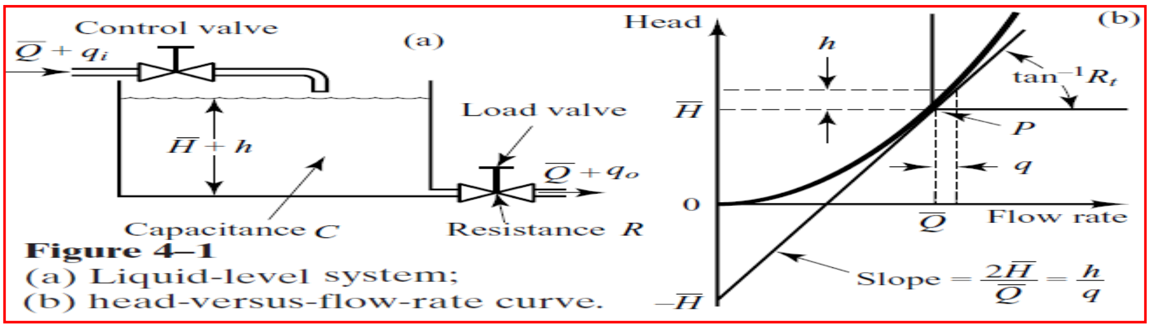


Figure 4–1
(a) Liquid-level system;
(b) head-versus-flow-rate curve.

where Q = steady-state liquid flow rate, m^3/sec
K = coefficient, m^2/sec $\quad H$ = steady-state head, m

For laminar flow, the resistance R_l is obtained as

$$R_l = \frac{dH}{dQ} = \frac{H}{Q}$$

The laminar-flow resistance is constant and is analogous to the electrical resistance.

If the flow through the restriction is turbulent, the steady-state flow rate is given by

$$Q = K\sqrt{H} \tag{4–1}$$

where Q = steady-state liquid flow rate, m^3/sec
K = coefficient, $\text{m}^{2.5}/\text{sec}$ $\quad H$ = steady-state head, m

The resistance R_t for turbulent flow is obtained from $\quad R_t = \dfrac{dH}{dQ}$

Since from Equation (4–1) we obtain $\quad dQ = \dfrac{K}{2\sqrt{H}}\, dH$

we have $\quad \dfrac{dH}{dQ} = \dfrac{2\sqrt{H}}{K} = \dfrac{2\sqrt{H}\sqrt{H}}{Q} = \dfrac{2H}{Q} \quad$ Thus, $\quad R_t = \dfrac{2H}{Q}$

The value of the turbulent-flow resistance R_t depends on the flow rate and the head. The value of R_t, however, may be considered constant if the changes in head and flow rate are small. By use of the turbulent-flow resistance, the relationship between Q and H can be given by

$$Q = \frac{2H}{R_t}$$

Such linearization is valid, provided that changes in the head and flow rate from their respective steady-state values are small.

In many practical cases, the value of the coefficient K in Equation (4–1), which depends on the flow coefficient and the area of restriction, is not known. Then the resistance may be determined by plotting the head-versus-flow-rate curve based on experimental data and measuring the slope of the curve at the operating condition. An example of such a plot is shown in Figure 4–1(b). In the figure, point P is the steady-state operating point. The tangent line to the curve at point P intersects the ordinate at point $(0, -\bar{H})$. Thus, the slope of this tangent line is $2\bar{H}/\bar{Q}$. Since the resistance R_t at the operating point P is given by $2\bar{H}/\bar{Q}$, the resistance R_t is the slope of the curve at the operating point.

Consider the operating condition in the neighborhood of point P. Define a small deviation of the head from the steady-state value as h and the corresponding small change of the flow rate as q. Then the slope of the curve at point P can be given by

$$\text{Slope of curve at point } P = \frac{h}{q} = \frac{2\bar{H}}{\bar{Q}} = R_t$$

The linear approximation is based on the fact that the actual curve does not differ much from its tangent line if the operating condition does not vary too much.

The capacitance C of a tank is defined to be the change in quantity of stored liquid necessary to cause a unit change in the potential (head). (The potential is the quantity that indicates the energy level of the system.)

$$C = \frac{\text{change in liquid stored, m}^3}{\text{change in head, m}}$$

It should be noted that the capacity (m^3) and the capacitance (m^2) are different. The capacitance of the tank is equal to its cross-sectional area. If this is constant, the capacitance is constant for any head.

Liquid-Level Systems. Consider the system shown in Figure 4–1(a). The variables are defined as follows:

$\bar{Q}$ = steady-state flow rate (before any change has occurred), m^3/sec
q_i = small deviation of inflow rate from its steady-state value, m^3/sec
q_o = small deviation of outflow rate from its steady-state value, m^3/sec
$\bar{H}$ = steady-state head (before any change has occurred), m
h = small deviation of head from its steady-state value, m

As stated previously, a system can be considered linear if the flow is laminar. Even if the flow is turbulent, the system can be linearized if changes in the variables are kept small. Based on the assumption that the system is either linear or linearized, the differential equation of this system can be obtained as follows: Since the inflow minus outflow during the small time interval dt is equal to the additional amount stored in the tank, we see that

$$C\, dh = (q_i - q_o)\, dt$$

From definition of resistance, the relationship between q_o and h is given by $\quad q_o = \dfrac{h}{R}$

The differential equation for this system for a constant value of R becomes

$$RC\frac{dh}{dt} + h = Rq_i \tag{4–2}$$

Note that RC is the time constant of the system. Taking the Laplace transforms of both sides of Equation (4–2), assuming the zero initial condition, we obtain

$(RCs + 1)H(s) = RQ_i(s) \quad$ where $\quad H(s) = \mathscr{L}[h] \quad$ and $\quad Q_i(s) = \mathscr{L}[q_i]$

If q_i is considered the input and h the output, the transfer function of the system is

$$\frac{H(s)}{Q_i(s)} = \frac{R}{RCs + 1}$$

If, however, q_o is taken as the output, the input being the same, then the transfer function is $\quad \dfrac{Q_o(s)}{Q_i(s)} = \dfrac{1}{RCs + 1} \quad$ where we have used the relationship $\quad Q_o(s) = \dfrac{1}{R}H(s)$

Liquid-Level Systems with Interaction. Consider the system shown in Figure 4–2. In this system, the two tanks interact. Thus the transfer function of the system is not the product of two first-order transfer functions.

In the following, we shall assume only small variations of the variables from the steady-state values. Using the symbols as defined in Figure 4–2, we can obtain the following equations for this system:

$\dfrac{h_1 - h_2}{R_1} = q_1 \;\; (4\text{–}3) \qquad C_1\dfrac{dh_1}{dt} = q - q_1 \;\; (4\text{–}4) \qquad \dfrac{h_2}{R_2} = q_2 \;\; (4\text{–}5) \qquad C_2\dfrac{dh_2}{dt} = q_1 - q_2 \;\; (4\text{–}6)$

If q is considered the input and q_2 the output, the transfer function of the system is

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1} \tag{4–7}$$

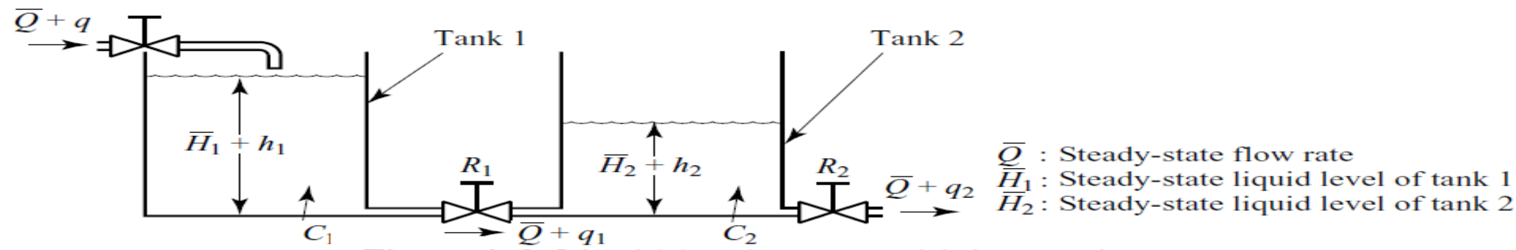


Figure 4–2 Liquid-level system with interaction.

It is instructive to obtain Equation (4–7), the transfer function of the interacted system, by block diagram reduction. From Equations (4–3) through (4–6), we obtain the elements of the block diagram, as shown in Figure 4–3(a). By connecting signals properly, we can construct a block diagram, as shown in Figure 4–3(b). This block diagram can be simplified, as shown in Figure 4–3(c). Further simplifications result in Figures 4–3(d) and (e). Figure 4–3(e) is equivalent to Equation (4–7).

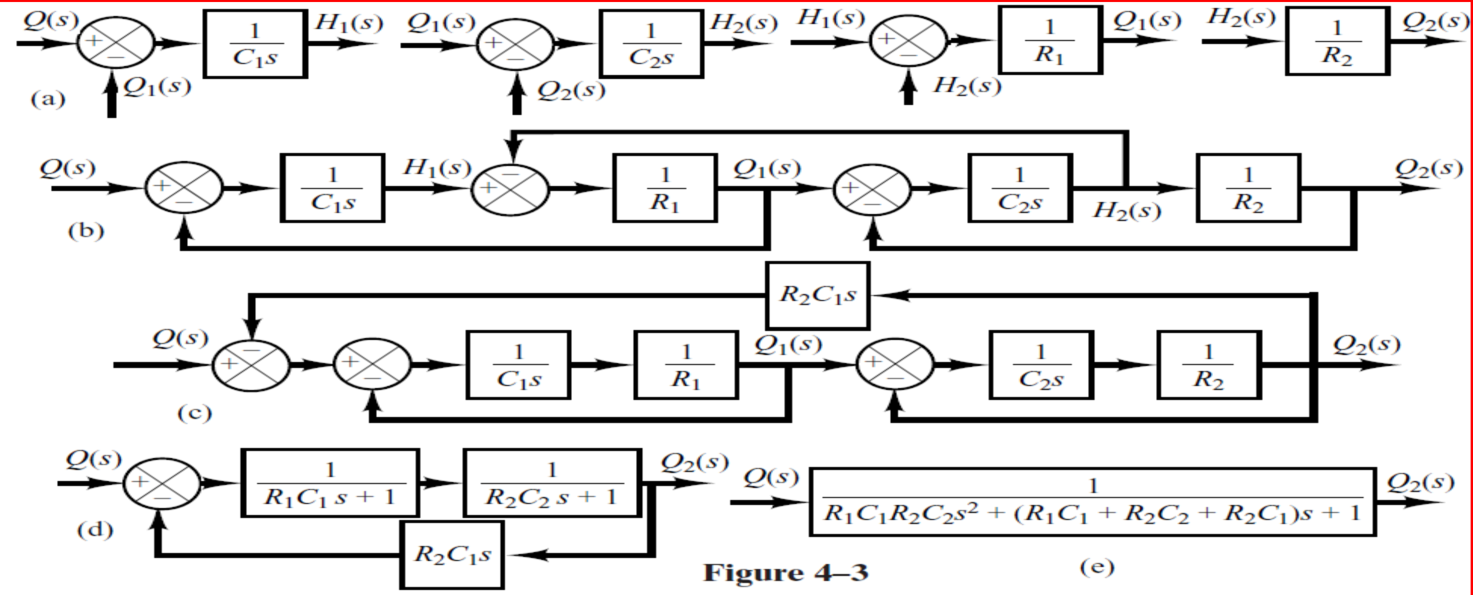


Figure 4–3

(a) Elements of block diagram of system shown in Figure 4–2; (b) block diagram of the system; (c)–(e) successive reductions of the block diagram.

Notice the similarity and difference between the transfer function given by Equation (4–7) and that given by Equation (3–33). The term R_2C_1s that appears in the denominator of Equation (4–7) exemplifies the interaction between the two tanks. Similarly, the term R_1C_2s in the denominator of Equation (3–33) represents the interaction between the two *RC* circuits shown in Figure 3–8.

4–3 PNEUMATIC SYSTEMS

In industrial applications pneumatic systems and hydraulic systems are frequently compared. Therefore, before we discuss pneumatic systems in detail, we shall give a brief comparison of these two kinds of systems.

Comparison Between Pneumatic Systems and Hydraulic Systems. The fluid generally found in pneumatic systems is air; in hydraulic systems it is oil. And it is primarily the different properties of the fluids involved that characterize the differences between the two systems. These differences can be listed as follows:

1. Air and gases are compressible, whereas oil is incompressible (except at high pressure).
2. Air lacks lubricating property and always contains water vapor. Oil functions as a hydraulic fluid as well as a lubricator.
3. The normal operating pressure of pneumatic systems is very much lower than that of hydraulic systems.
4. Output powers of pneumatic systems are considerably less than those of hydraulic systems.
5. Accuracy of pneumatic actuators is poor at low velocities, whereas accuracy of hydraulic actuators may be made satisfactory at all velocities.
6. In pneumatic systems, external leakage is permissible to a certain extent, but internal leakage must be avoided because the effective pressure difference is rather small. In hydraulic systems internal leakage is permissible to a certain extent, but external leakage must be avoided.
7. No return pipes are required in pneumatic systems when air is used, whereas they are always needed in hydraulic systems.
8. Normal operating temperature for pneumatic systems is 5° to 60°C (41° to 140°F). The pneumatic system, however, can be operated in the 0° to 200°C (32° to 392°F) range. Pneumatic systems are insensitive to temperature changes, in contrast to hydraulic systems, in which fluid friction due to viscosity depends greatly on temperature. Normal operating temperature for hydraulic systems is 20° to 70°C (68° to 158°F).
9. Pneumatic systems are fire- and explosion-proof, whereas hydraulic systems are not, unless nonflammable liquid is used.

In what follows we begin with a mathematical modeling of pneumatic systems. Then we shall present pneumatic proportional controllers.

We shall first give detailed discussions of the principle by which proportional controllers operate. Then we shall treat methods for obtaining derivative and integral control actions. Throughout the discussions, we shall place emphasis on the fundamental principles, rather than on the details of the operation of the actual mechanisms.

Pneumatic Systems. The past decades have seen a great development in low-pressure pneumatic controllers for industrial control systems, and today they are used extensively in industrial processes. Reasons for their broad appeal include an explosion-proof character, simplicity, and ease of maintenance.

Resistance and Capacitance of Pressure Systems. Many industrial processes and pneumatic controllers involve the flow of a gas or air through connected pipelines and pressure vessels.

Consider the pressure system shown in Figure 4–4(a). The gas flow through the restriction is a function of the gas pressure difference $p_i - p_o$. Such a pressure system may be characterized in terms of a resistance and a capacitance.

The gas flow resistance R may be defined as follows:

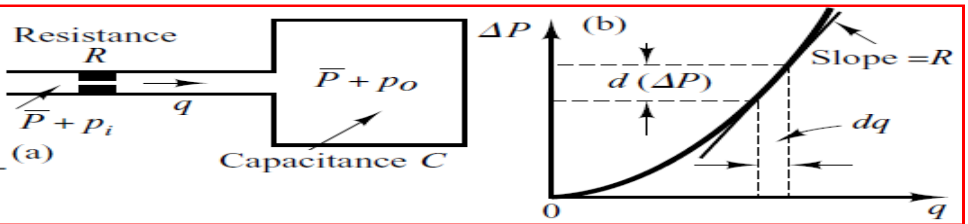
$$R = \frac{\text{change in gas pressure difference, lb}_f/\text{ft}^2}{\text{change in gas flow rate, lb/sec}} \quad \text{or} \quad R = \frac{d(\Delta P)}{dq} \qquad (4\text{–}8)$$

where $d(\Delta P)$ is a small change in the gas pressure difference and dq is a small change in the gas flow rate. Computation of the value of the gas flow resistance R may be quite time consuming. Experimentally, however, it can be easily determined from a plot of the pressure difference versus flow rate by calculating the slope of the curve at a given operating condition, as shown in Figure 4–4(b).

The capacitance of the pressure vessel may be defined by

$$C = \frac{\text{change in gas stored, lb}}{\text{change in gas pressure, lb}_f/\text{ft}^2} \quad \text{or} \quad C = \frac{dm}{dp} = V\frac{d\rho}{dp} \qquad (4\text{–}9)$$

Figure 4–4
(a) Schematic diagram of a pressure system; (b) pressure-difference-versus-flow-rate curve.



where C = capacitance, lb-ft^2/lb$_f$ $\quad m$ = mass of gas in vessel, lb $\quad \rho$ = density, lb/ft^3
p = gas pressure, lb$_f$/ft^2 $\quad V$ = volume of vessel, ft^3

The capacitance of the pressure system depends on the type of expansion process involved. The capacitance can be calculated by use of the ideal gas law. If the gas expansion process is polytropic and the change of state of the gas is between isothermal and adiabatic, then

$$p\left(\frac{V}{m}\right)^n = \frac{p}{\rho^n} = \text{constant} = K \qquad (4\text{–}10)$$

where n = polytropic exponent.

For ideal gases, $\quad p\bar{v} = \bar{R}T \quad$ or $\quad pv = \frac{\bar{R}}{M}T$

where p = absolute pressure, lb$_f$/ft^2 $\quad \bar{R}$ = universal gas constant, ft-lb$_f$/lb-mole °R
$\bar{v}$ = volume occupied by 1 mole of a gas, ft^3/lb-mole $\quad T$ = absolute temperature, °R
v = specific volume of gas, ft^3/lb $\quad M$ = molecular weight of gas per mole, lb/lb-mole

Thus $pv = \frac{p}{\rho} = \frac{\bar{R}}{M}T = R_{gas}T$ $\quad$ where R_{gas} = gas constant, ft-lb$_f$/lb °R. $\qquad$ (4–11)

The polytropic exponent n is unity for isothermal expansion. For adiabatic expansion, n is equal to the ratio of specific heats c_p/c_v, where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. In many practical cases, the value of n is approximately constant, and thus the capacitance may be considered constant.

The value of $d\rho/dp$ is obtained from Equations (4–10) and (4–11). From Equation (4–10) we have

$$dp = Kn\rho^{n-1}\,d\rho \quad \text{or} \quad \frac{d\rho}{dp} = \frac{1}{Kn\rho^{n-1}} = \frac{\rho^n}{pn\rho^{n-1}} = \frac{\rho}{pn}$$

Substituting Equation (4–11) into this last equation, we get $\quad d\rho/dp = 1/nR_{gas}T$

The capacitance C is then obtained as

$$C = \frac{V}{nR_{gas}T} \qquad (4\text{–}12)$$

The capacitance of a given vessel is constant if the temperature stays constant. (In many practical cases, the polytropic exponent n is approximately 1.0 ~ 1.2 for gases in uninsulated metal vessels.)

Pressure Systems. Consider the system shown in Figure 4–4(a). If we assume only small deviations in the variables from their respective steady-state values, then this system may be considered linear. Let us define

$\bar{P}$ = gas pressure in vessel at steady state (before changes in pressure occurred), lb$_f$/ft^2
p_i = small change in inflow gas pressure, lb$_f$/ft^2 $\quad m$ = mass of gas in the vessel, lb
p_o = small change in gas pressure in the vessel, lb$_f$/ft^2 $\quad q$ = gas flow rate, lb/sec
V = volume of the vessel, ft^3 $\quad \rho$ = density of gas, lb/ft^3

For small values of p_i and p_o, the resistance R given by Equation (4–8) becomes constant and may be written as

$$R = \frac{p_i - p_o}{q}$$

The capacitance C is given by Equation (4–9), or $\quad C = \frac{dm}{dp}$

Since the pressure change dp_o times the capacitance C is equal to the gas added to the vessel during dt seconds, we obtain

$$C\,dp_o = q\,dt \quad \text{or} \quad C\frac{dp_o}{dt} = \frac{p_i - p_o}{R} \quad \text{which can be written as} \quad RC\frac{dp_o}{dt} + p_o = p_i$$

If p_i and p_o are considered the input and output, respectively, then the transfer function of the system is

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs + 1}$$

where RC has the dimension of time and is the time constant of the system.

Pneumatic Nozzle–Flapper Amplifiers. A schematic diagram of a pneumatic nozzle–flapper amplifier is shown in Figure 4–5(a). The power source for this amplifier is a supply of air at constant pressure. The nozzle–flapper amplifier converts small changes in the position of the flapper into large changes in the back pressure in the nozzle. Thus a large power output can be controlled by the very little power that is needed to position the flapper.

In Figure 4–5(a), pressurized air is fed through the orifice, and the air is ejected from the nozzle toward the flapper. Generally, the supply pressure P_s for such a controller is 20 psig (1.4 kg_f/cm^2 gage). The diameter of the orifice is on the order of 0.01 in. (0.25 mm) and that of the nozzle is on the order of 0.016 in. (0.4 mm). To ensure proper functioning of the amplifier, the nozzle diameter must be larger than the orifice diameter.

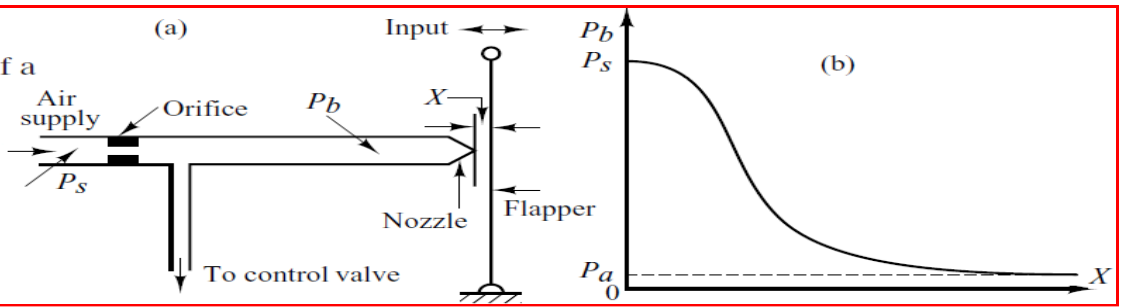
In operating this system, the flapper is positioned against the nozzle opening. The nozzle back pressure P_b is controlled by the nozzle–flapper distance X. As the flapper approaches the nozzle, the opposition to the flow of air through the nozzle increases, with the result that the nozzle back pressure P_b increases. If the nozzle is completely closed by the flapper, the nozzle back pressure P_b becomes equal to the supply pressure P_s. If the flapper is moved away from the nozzle, so that the nozzle–flapper distance is wide (on the order of 0.01 in.), then there is practically no restriction to flow, and the nozzle back pressure P_b takes on a minimum value that depends on the nozzle–flapper device. (The lowest possible pressure will be the ambient pressure P_a.)

Note that, because the air jet puts a force against the flapper, it is necessary to make the nozzle diameter as small as possible.

A typical curve relating the nozzle back pressure P_b to the nozzle–flapper distance X is shown in Figure 4–5(b). The steep and almost linear part of the curve is utilized in the actual operation of the nozzle–flapper amplifier. Because the range of flapper displacements is restricted to a small value, the change in output pressure is also small, unless the curve is very steep.

The nozzle–flapper amplifier converts displacement into a pressure signal. Since industrial process control systems require large output power to operate large pneumatic actuating valves, the power amplification of the nozzle–flapper amplifier is usually insufficient. Consequently, a pneumatic relay is often needed as a power amplifier in connection with the nozzle–flapper amplifier.

Figure 4–5
(a) Schematic diagram of a pneumatic nozzle–flapper amplifier; (b) characteristic curve relating nozzle back pressure and nozzle–flapper distance.



Pneumatic Relays. In practice, in a pneumatic controller, a nozzle–flapper amplifier acts as the first-stage amplifier and a pneumatic relay as the second-stage amplifier. The pneumatic relay is capable of handling a large quantity of airflow.

A schematic diagram of a pneumatic relay is shown in Figure 4–6(a). As the nozzle back pressure P_b increases, the diaphragm valve moves downward. The opening to the atmosphere decreases and the opening to the pneumatic valve increases, thereby increasing the control pressure P_c. When the diaphragm valve closes the opening to the atmosphere, the control pressure P_c becomes equal to the supply pressure P_s. When the nozzle back pressure P_b decreases and the diaphragm valve moves upward and shuts off the air supply, the control pressure P_c drops to the ambient pressure P_a. The control pressure P_c can thus be made to vary from 0 psig to full supply pressure, usually 20 psig.

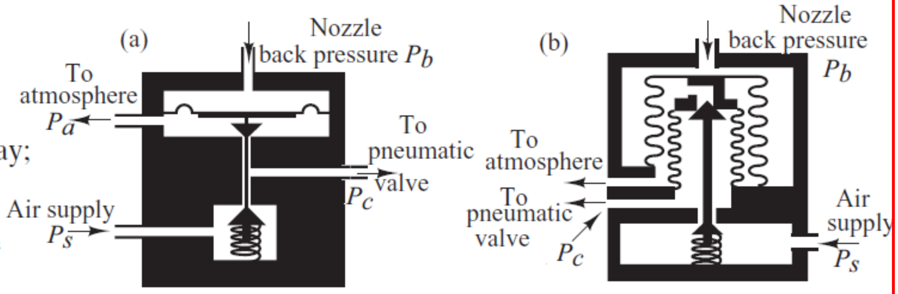
The total movement of the diaphragm valve is very small. In all positions of the valve, except at the position to shut off the air supply, air continues to bleed into the atmosphere, even after the equilibrium condition is attained between the nozzle back pressure and the control pressure. Thus the relay shown in Figure 4–6(a) is called a bleed-type relay.

There is another type of relay, the nonbleed type. In this one the air bleed stops when the equilibrium condition is obtained and, therefore, there is no loss of pressurized air at steady-state operation. Note, however, that the nonbleed-type relay must have an atmospheric relief to release the control pressure P_c from the pneumatic actuating valve. A schematic diagram of a nonbleed-type relay is shown in Figure 4–6(b).

In either type of relay, the air supply is controlled by a valve, which is in turn controlled by the nozzle back pressure. Thus, the nozzle back pressure is converted into the control pressure with power amplification.

Since the control pressure P_c changes almost instantaneously with changes in the nozzle back pressure P_b, the time constant of the pneumatic relay is negligible compared with the other larger time constants of the pneumatic controller and the plant.

Figure 4–6 (a) Schematic diagram of a bleed-type relay; (b) schematic diagram of a nonbleed-type relay.



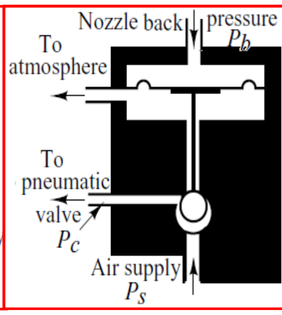


Figure 4–7 Reverse-acting relay.

It is noted that some pneumatic relays are reverse acting. For example, the relay shown in Figure 4–7 is a reverse-acting relay. Here, as the nozzle back pressure P_b increases, the ball valve is forced toward the lower seat, thereby decreasing the control pressure P_c. Thus, this relay is a reverse-acting relay.

Pneumatic Proportional Controllers (Force-Distance Type). Two types of pneumatic controllers, one called the force-distance type and the other the force-balance type, are used extensively in industry. Regardless of how differently industrial pneumatic controllers may appear, careful study will show the close similarity in the functions of the pneumatic circuit. Here we shall consider the force-distance type of pneumatic controllers.

Figure 4–8(a) shows a schematic diagram of such a proportional controller. The nozzle–flapper amplifier constitutes the first-stage amplifier, and the nozzle back pressure is controlled by the nozzle–flapper distance. The relay-type amplifier constitutes the second-stage amplifier. The nozzle back pressure determines the position of the diaphragm valve for the second-stage amplifier, which is capable of handling a large quantity of airflow.

In most pneumatic controllers, some type of pneumatic feedback is employed. Feedback of the pneumatic output reduces the amount of actual movement of the flapper. Instead of mounting the flapper on a fixed point, as shown in Figure 4–8(b), it is often pivoted on the feedback bellows, as shown in Figure 4–8(c). The amount of feedback can be regulated by introducing a variable linkage between the feedback bellows and the flapper connecting point. The flapper then becomes a floating link. It can be moved by both the error signal and the feedback signal.

The operation of the controller shown in Figure 4–8(a) is as follows. The input signal to the two-stage pneumatic amplifier is the actuating error signal. Increasing the actuating error signal moves the flapper to the left. This move will, in turn, increase the nozzle back pressure, and the diaphragm valve moves downward. This results in an increase of the control pressure. This increase will cause bellows F to expand and move the flapper to the right, thus opening the nozzle. Because of this feedback, the nozzle–flapper displacement is very small, but the change in the control pressure can be large.

It should be noted that proper operation of the controller requires that the feedback bellows move the flapper less than that movement caused by the error signal alone. (If these two movements were equal, no control action would result.)

Equations for this controller can be derived as follows. When the actuating error is zero, or $e = 0$, an equilibrium state exists with the nozzle–flapper distance equal to $\bar{X}$, the

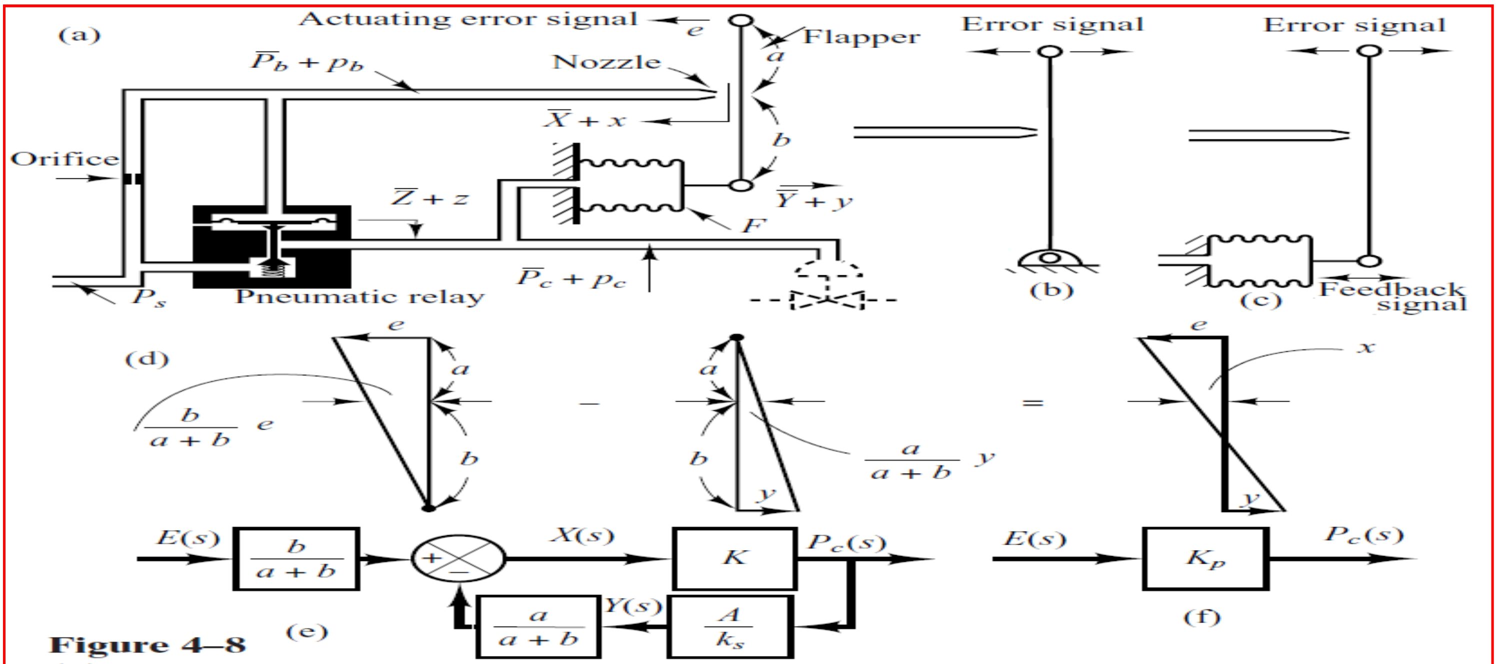


Figure 4–8
(a) Schematic of a force-distance type of pneumatic proportional controller; (b) flapper mounted on fixed point; (c) flapper mounted on feedback bellows; (d) displacement x as a result of addition of two small displacements; (e) block diagram for controller; (f) simplified block diagram for controller.

displacement of bellows equal to $\bar{Y}$, the displacement of the diaphragm equal to $\bar{Z}$, the nozzle back pressure equal to $\bar{P}_b$, and the control pressure equal to $\bar{P}_c$. When an actuating error exists, the nozzle–flapper distance, the displacement of the bellows, the displacement of the diaphragm, the nozzle back pressure, and the control pressure deviate from their respective equilibrium values. Let these deviations be x, y, z, p_b, and p_c, respectively. (The positive direction for each displacement variable is indicated by an arrowhead in the diagram.)

Assuming that the relationship between the variation in the nozzle back pressure and the variation in the nozzle–flapper distance is linear, we have

$$p_b = K_1 x \tag{4–13}$$

where K_1 is a positive constant. For the diaphragm valve,

$$p_b = K_2 z \tag{4–14}$$

where K_2 is a positive constant. The position of the diaphragm valve determines the control pressure. If the diaphragm valve is such that the relationship between p_c and z is linear, then

$$p_c = K_3 z \tag{4–15}$$

where K_3 is a positive constant. From Equations (4–13), (4–14), and (4–15), we obtain

$$p_c = \frac{K_3}{K_2} p_b = \frac{K_1 K_3}{K_2} x = Kx \tag{4–16}$$

where $K = K_1K_3/K_2$ is a positive constant. For the flapper, since there are two small movements (e and y) in opposite directions, we can consider such movements separately and add up the results of two movements into one displacement x. See Figure 4–8(d). Thus, for the flapper movement, we have

$$x = \frac{b}{a+b} e - \frac{a}{a+b} y \tag{4–17}$$

The bellows acts like a spring, and the following equation holds true:

$$Ap_c = k_s y \tag{4–18}$$

where A is the effective area of the bellows and k_s is the equivalent spring constant—that is, the stiffness due to the action of the corrugated side of the bellows.

Assuming that all variations in the variables are within a linear range, we can obtain a block diagram for this system from Equations (4–16), (4–17), and (4–18) as shown in Figure 4–8(e). From Figure 4–8(e), it can be clearly seen that the pneumatic controller shown in Figure 4–8(a) itself is a feedback system. The transfer function between p_c and e is given by

$$\frac{P_c(s)}{E(s)} = \frac{b/(a+b)\ K}{1+K\ \ a/(a+b)(A/k_s)} = K_p \tag{4–19}$$

A simplified block diagram is shown in Figure 4–8(f). Since p_c and e are proportional, the pneumatic controller shown in Figure 4–8(a) is a *pneumatic proportional controller*. As seen from Equation (4–19), the gain of the pneumatic proportional controller can be widely varied by adjusting the flapper connecting linkage. [The flapper connecting linkage is not shown in Figure 4–8(a).] In most commercial proportional controllers an adjusting knob or other mechanism is provided for varying the gain by adjusting this linkage.

As noted earlier, the actuating error signal moved the flapper in one direction, and the feedback bellows moved the flapper in the opposite direction, but to a smaller degree.

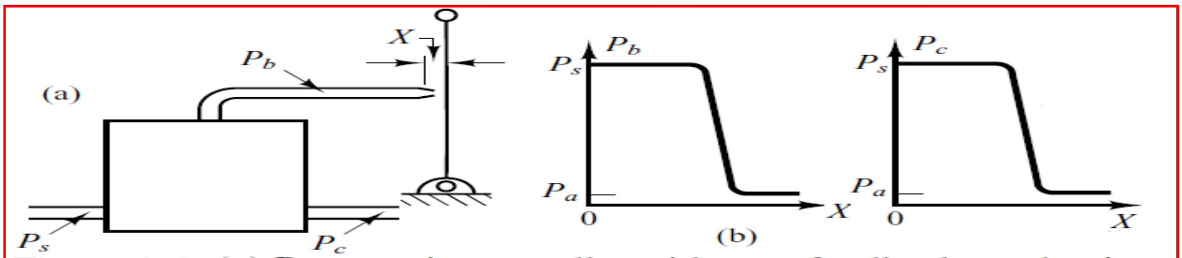


Figure 4–9 (a) Pneumatic controller without a feedback mechanism; (b) curves P_b versus X and P_c versus X.

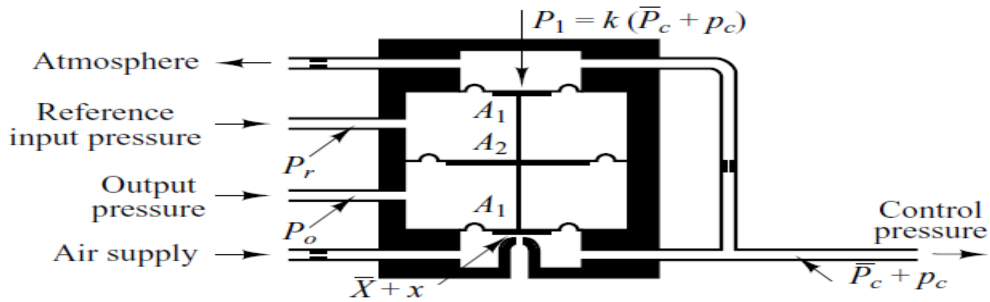
The effect of the feedback bellows is thus to reduce the sensitivity of the controller. The principle of feedback is commonly used to obtain wide proportional-band controllers.

Pneumatic controllers that do not have feedback mechanisms [which means that one end of the flapper is fixed, as shown in Figure 4–9(a)] have high sensitivity and are called *pneumatic two-position controllers* or *pneumatic on–off controllers*. In such a controller, only a small motion between the nozzle and the flapper is required to give a complete change from the maximum to the minimum control pressure. The curves relating P_b to X and P_c to X are shown in Figure 4–9(b). Notice that a small change in X can cause a large change in P_b, which causes the diaphragm valve to be completely open or completely closed.

Pneumatic Proportional Controllers (Force-Balance Type). Figure 4–10 shows a schematic diagram of a force-balance type pneumatic proportional controller. Force-balance type controllers are in extensive use in industry. Such controllers are called stack controllers. The basic principle of operation does not differ from that of the force-distance type controller. The main advantage of the force-balance type controller is that it eliminates many mechanical linkages and pivot joints, thereby reducing the effects of friction.

In what follows, we shall consider the principle of the force-balance type controller. In the controller shown in Figure 4–10, the reference input pressure P_r and the output pressure P_o are fed to large diaphragm chambers. Note that a force-balance type pneumatic controller operates only on pressure signals. Therefore, it is necessary to convert the reference input and system output to corresponding pressure signals.

Figure 4–10 Schematic diagram of a force-balance type pneumatic proportional controller.



As in the case of the force-distance type controller, this controller employs a flapper, nozzle, and orifices. In Figure 4–10, the drilled opening in the bottom chamber is the nozzle. The diaphragm just above the nozzle acts as a flapper.

The operation of the force-balance type controller shown in Figure 4–10 may be summarized as follows: 20-psig air from an air supply flows through an orifice, causing a reduced pressure in the bottom chamber. Air in this chamber escapes to the atmosphere through the nozzle. The flow through the nozzle depends on the gap and the pressure drop across it. An increase in the reference input pressure P_r, while the output pressure P_o remains the same, causes the valve stem to move down, decreasing the gap between the nozzle and the flapper diaphragm. This causes the control pressure P_c to increase. Let

$$p_e = P_r - P_o \tag{4–20}$$

If $p_e = 0$, there is an equilibrium state with the nozzle–flapper distance equal to $\bar{X}$ and the control pressure equal to $\bar{P}_c$. At this equilibrium state, $P_1 = \bar{P}_c k$ (where $k < 1$) and

$$\bar{X} = \alpha(\bar{P}_c A_1 - \bar{P}_c k A_1) \tag{4–21}$$

where α is a constant.

Let us assume that $p_e \neq 0$ and define small variations in the nozzle–flapper distance and control pressure as x and p_c, respectively. Then we obtain the following equation:

$$\bar{X} + x = \alpha[(\bar{P}_c + p_c)A_1 - (\bar{P}_c + p_c)kA_1 - p_e(A_2 - A_1)] \tag{4–22}$$

From Equations (4–21) and (4–22), we obtain

$$x = \alpha[p_c(1 - k)A_1 - p_e(A_2 - A_1)] \tag{4–23}$$

At this point, we must examine the quantity x. In the design of pneumatic controllers, the nozzle–flapper distance is made quite small. In view of the fact that x/α is very much smaller than $p_c(1 - k)A_1$ or $p_e(A_2 - A_1)$—that is, for $p_e \neq 0$

$$\frac{x}{\alpha} \ll p_c(1 - k)A_1 \qquad \frac{x}{\alpha} \ll p_e(A_2 - A_1)$$

we may neglect the term x in our analysis. Equation (4–23) can then be rewritten to reflect this assumption as follows: $p_c(1 - k)A_1 = p_e(A_2 - A_1)$

and the transfer function between p_c and p_e becomes $\dfrac{P_c(s)}{P_e(s)} = \dfrac{A_2 - A_1}{A_1}\dfrac{1}{1 - k} = K_p$

where p_e is defined by Equation (4–20). The controller shown in Figure 4–10 is a proportional controller. The value of gain K_p increases as k approaches unity. Note that the value of k depends on the diameters of the orifices in the inlet and outlet pipes of the feedback chamber. (The value of k approaches unity as the resistance to flow in the orifice of the inlet pipe is made smaller.)

Pneumatic Actuating Valves. One characteristic of pneumatic controls is that they almost exclusively employ pneumatic actuating valves. A pneumatic actuating valve can provide a large power output. (Since a pneumatic actuator requires a large power input to produce a large power output, it is necessary that a sufficient quantity of pressurized air be available.) In practical pneumatic actuating valves, the valve characteristics may not be linear; that is, the flow may not be directly proportional to the valve stem position, and also there may be other nonlinear effects, such as hysteresis.

Consider the schematic diagram of a pneumatic actuating valve shown in Figure 4–11. Assume that the area of the diaphragm is A. Assume also that when the actuating error is zero, the control pressure is equal to $\bar{P}_c$ and the valve displacement is equal to $\bar{X}$.

In the following analysis, we shall consider small variations in the variables and linearize the pneumatic actuating valve. Let us define the small variation in the control pressure and the corresponding valve displacement to be p_c and x, respectively. Since a small change in the pneumatic pressure force applied to the diaphragm repositions the load, consisting of the spring, viscous friction, and mass, the force-balance equation becomes

$$Ap_c = m\ddot{x} + b\dot{x} + kx$$

where m = mass of the valve and valve stem $\quad b$ = viscous-friction coefficient
k = spring constant

If the force due to the mass and viscous friction are negligibly small, then this last equation can be simplified to

$$Ap_c = kx$$

The transfer function between x and p_c thus becomes $\dfrac{X(s)}{P_c(s)} = \dfrac{A}{k} = K_c$

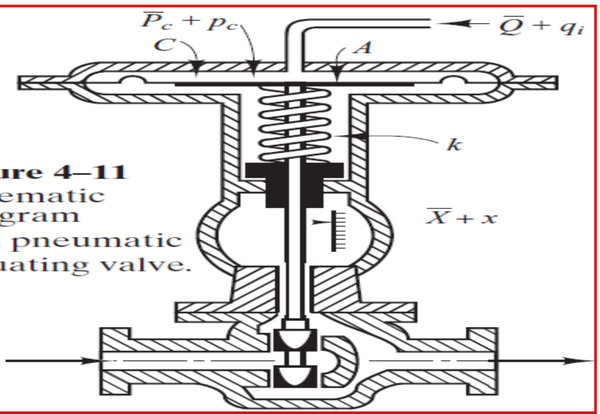


Figure 4–11 Schematic diagram of a pneumatic actuating valve.

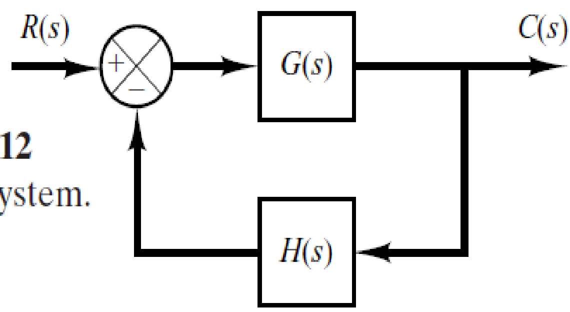


Figure 4–12 Control system.

where $X(s) = \mathscr{L}[x]$ and $P_c(s) = \mathscr{L}[p_c]$. If q_i, the change in flow through the pneumatic actuating valve, is proportional to x, the change in the valve-stem displacement, then

$$\frac{Q_i(s)}{X(s)} = K_q$$

where $Q_i(s) = \mathscr{L}[q_i]$ and K_q is a constant. The transfer function between q_i and p_c becomes

$$\frac{Q_i(s)}{P_c(s)} = K_c K_q = K_v$$

where K_v is a constant.

The standard control pressure for this kind of a pneumatic actuating valve is between 3 and 15 psig. The valve-stem displacement is limited by the allowable stroke of the diaphragm and is only a few inches. If a longer stroke is needed, a piston–spring combination may be employed.

In pneumatic actuating valves, the static-friction force must be limited to a low value so that excessive hysteresis does not result. Because of the compressibility of air, the control action may not be positive; that is, an error may exist in the valve-stem position. The use of a valve positioner results in improvements in the performance of a pneumatic actuating valve.

Basic Principle for Obtaining Derivative Control Action. We shall now present methods for obtaining derivative control action. We shall again place the emphasis on the principle and not on the details of the actual mechanisms.

The basic principle for generating a desired control action is to insert the inverse of the desired transfer function in the feedback path. For the system shown in Figure 4–12, the closed-loop transfer function is $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

If $|G(s)H(s)| \gg 1$, then $C(s)/R(s)$ can be modified to $\frac{C(s)}{R(s)} = \frac{1}{H(s)}$

Thus, if proportional-plus-derivative control action is desired, we insert an element having the transfer function $1/(Ts + 1)$ in the feedback path.

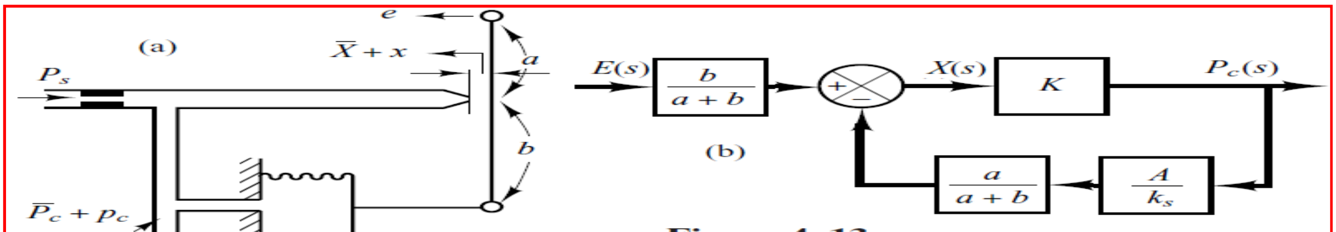


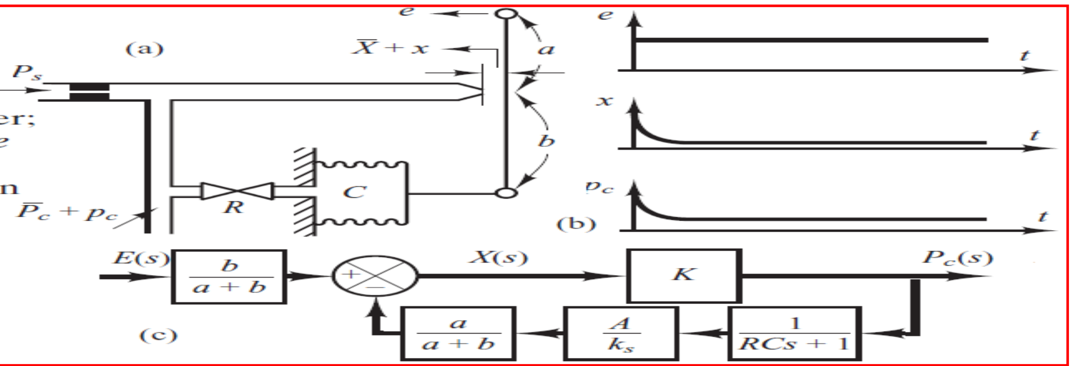
Figure 4–13
(a) Pneumatic proportional controller; (b) block diagram of the controller.

Consider the pneumatic controller shown in Figure 4–13(a). Considering small changes in the variables, we can draw a block diagram of this controller as shown in Figure 4–13(b). From the block diagram we see that the controller is of proportional type.

We shall now show that the addition of a restriction in the negative feedback path will modify the proportional controller to a proportional-plus-derivative controller, or a PD controller.

Consider the pneumatic controller shown in Figure 4–14(a). Assuming again small changes in the actuating error, nozzle–flapper distance, and control pressure, we can summarize the operation of this controller as follows: Let us first assume a small step change in e.

Figure 4–14
(a) Pneumatic proportional-plus-derivative controller; (b) step change in e and the corresponding changes in x and p_c plotted versus t; (c) block diagram of the controller.



Then the change in the control pressure p_c will be instantaneous. The restriction R will momentarily prevent the feedback bellows from sensing the pressure change p_c. Thus the feedback bellows will not respond momentarily, and the pneumatic actuating valve will feel the full effect of the movement of the flapper. As time goes on, the feedback bellows will expand. The change in the nozzle–flapper distance x and the change in the control pressure p_c can be plotted against time t, as shown in Figure 4–14(b). At steady state, the feedback bellows acts like an ordinary feedback mechanism. The curve p_c versus t clearly shows that this controller is of the proportional-plus-derivative type.

A block diagram corresponding to this pneumatic controller is shown in Figure 4–14(c). In the block diagram, K is a constant, A is the area of the bellows, and k_s is the equivalent spring constant of the bellows. The transfer function between p_c and e can be obtained from the block diagram as follows:

$$\frac{P_c(s)}{E(s)} = \frac{K\,b/(a+b)}{1 + Ka/(a+b)(A/k_s)(1/(RCs+1))}$$

In such a controller the loop gain $|KaA/[(a + b)k_s(RCs + 1)]|$ is made much greater than unity. Thus the transfer function $P_c(s)/E(s)$ can be simplified to give

$$\frac{P_c(s)}{E(s)} = K_p(1 + T_d s) \qquad \text{where} \quad K_p = \frac{bk_s}{aA}, \qquad T_d = RC$$

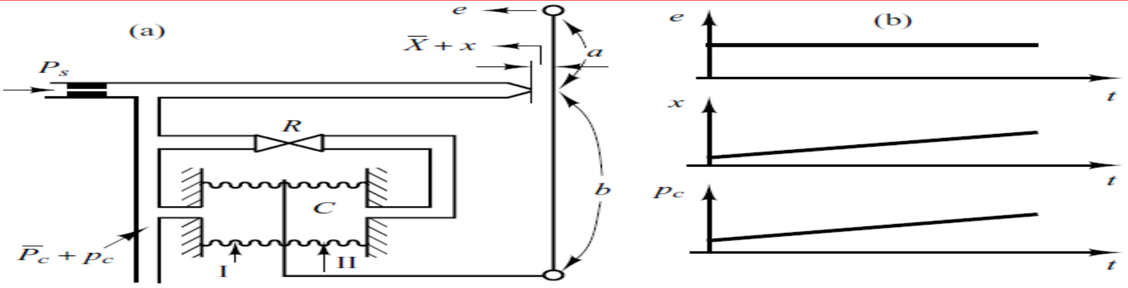
Thus, delayed negative feedback, or the transfer function $1/(RCs + 1)$ in the feedback path, modifies the proportional controller to a proportional-plus-derivative controller.

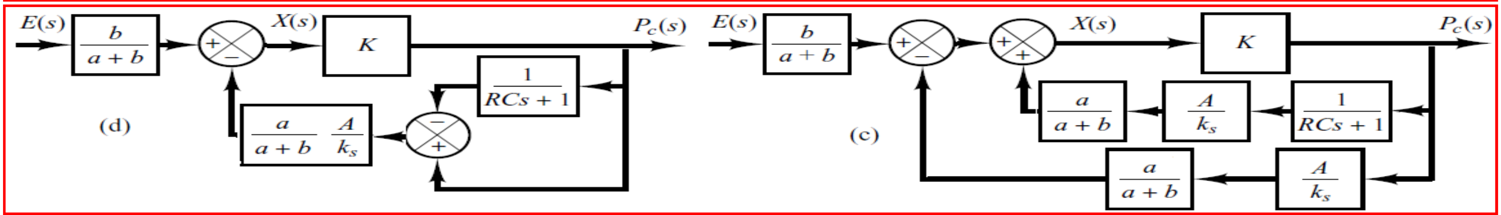
Note that if the feedback valve is fully opened, the control action becomes proportional. If the feedback valve is fully closed, the control action becomes narrow-band proportional (on–off).

Obtaining Pneumatic Proportional-Plus-Integral Control Action. Consider the proportional controller shown in Figure 4–13(a). Considering small changes in the variables, we can show that the addition of delayed positive feedback will modify this proportional controller to a proportional-plus-integral controller, or a PI controller.

Consider the pneumatic controller shown in Figure 4–15(a). The operation of this controller is as follows: The bellows denoted by I is connected to the control pressure source without any restriction. The bellows denoted by II is connected to the control pressure source through a restriction. Let us assume a small step change in the actuating error. This will cause the back pressure in the nozzle to change instantaneously. Thus a change in the control pressure p_c also occurs instantaneously. Due to the restriction of the valve in the path to bellows II, there will be a pressure drop across the valve. As time goes on, air will flow across the valve in such a way that the change in pressure in bellows II attains the value p_c. Thus bellows II will expand or contract as time elapses in such a way as to move the flapper an additional amount in the direction of the original displacement e. This will cause the back pressure p_c in the nozzle to change continuously, as shown in Figure 4–15(b).

Figure 4–15
(a) Pneumatic proportional-plus-integral controller;
(b) step change in e and corresponding changes in x and p_c plotted versus t;
(c) block diagram of the controller;
(d) simplified block diagram.





Note that the integral control action in the controller takes the form of slowly canceling the feedback that the proportional control originally provided.

A block diagram of this controller under the assumption of small variations in the variables is shown in Figure 4–15(c). A simplification of this block diagram yields Figure 4–15(d). The transfer function of this controller is

$$\frac{P_c(s)}{E(s)} = \frac{K\,b/(a+b)}{1 + (Ka/(a+b))(A/k_s)\ (1 - 1/(RCs+1))}$$

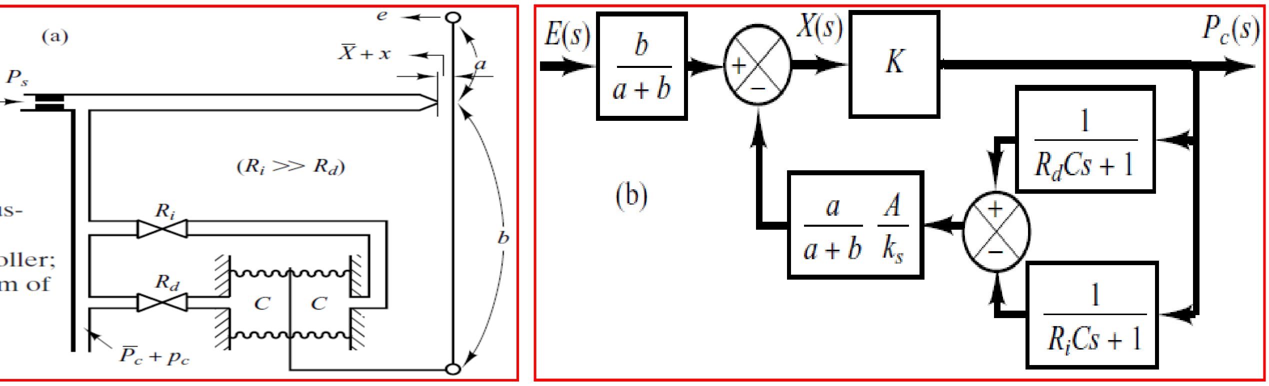
where K is a constant, A is the area of the bellows, and k_s is the equivalent spring constant of the combined bellows. If $|KaARCs/[(a+b)k_s(RCs+1)]| \gg 1$, which is usually the case, the transfer function can be simplified to

$$\frac{P_c(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right) \qquad \text{where} \qquad K_p = \frac{bk_s}{aA}, \qquad T_i = RC$$

Obtaining Pneumatic Proportional-Plus-Integral-Plus-Derivative Control Action

A combination of the pneumatic controllers shown in Figures 4–14(a) and 4–15(a) yields a proportional-plus-integral-plus-derivative controller, or a PID controller. Figure 4–16(a) shows a schematic diagram of such a controller. Figure 4–16(b) shows a block diagram of this controller under the assumption of small variations in the variables.

Figure 4–16
(a) Pneumatic proportional-plus-integral-plus-derivative controller;
(b) block diagram of the controller.



The transfer function of this controller is

$$\frac{P_c(s)}{E(s)} = \frac{bK/(a+b)}{1 + \dfrac{Ka}{a+b}\dfrac{A}{k_s}\dfrac{(R_iC - R_dC)s}{(R_dCs+1)(R_iCs+1)}}$$

By defining $T_i = R_iC$, $T_d = R_dC$

and noting that under normal operation $|KaA(T_i - T_d)s/[(a+b)k_s(T_ds+1)(T_is+1)]| \gg 1$ and $T_i \gg T_d$, we obtain

$$\frac{P_c(s)}{E(s)} \doteq \frac{bk_s}{aA}\frac{(T_ds+1)(T_is+1)}{(T_i - T_d)s} \doteq \frac{bk_s}{aA}\frac{T_dT_is^2 + T_is + 1}{T_is}$$

$$\frac{P_c(s)}{E(s)} = K_p\left(1 + \frac{1}{T_is} + T_ds\right) \qquad \text{where} \qquad K_p = \frac{bk_s}{aA} \qquad (4–24)$$

Equation (4–24) indicates that the controller shown in Figure 4–16(a) is a proportional-plus-integral-plus-derivative controller or a PID controller.

4–4 HYDRAULIC SYSTEMS

Except for low-pressure pneumatic controllers, compressed air has seldom been used for the continuous control of the motion of devices having significant mass under external load forces. For such a case, hydraulic controllers are generally preferred.

Hydraulic Systems. The widespread use of hydraulic circuitry in machine tool applications, aircraft control systems, and similar operations occurs because of such factors as positiveness, accuracy, flexibility, high horsepower-to-weight ratio, fast starting, stopping, and reversal with smoothness and precision, and simplicity of operations.

The operating pressure in hydraulic systems is somewhere between 145 and 5000 $lb_f/in.^2$ (between 1 and 35 MPa). In some special applications, the operating pressure may go up to 10,000 $lb_f/in.^2$ (70 MPa). For the same power requirement, the weight and size of the hydraulic unit can be made smaller by increasing the supply pressure. With high-pressure hydraulic systems, very large force can be obtained. Rapid-acting, accurate positioning of heavy loads is possible with hydraulic systems. A combination of electronic and hydraulic systems is widely used because it combines the advantages of both electronic control and hydraulic power.

Advantages and Disadvantages of Hydraulic Systems. There are certain advantages and disadvantages in using hydraulic systems rather than other systems. Some of the advantages are the following:

1. Hydraulic fluid acts as a lubricant, in addition to carrying away heat generated in the system to a convenient heat exchanger.
2. Comparatively small-sized hydraulic actuators can develop large forces or torques.
3. Hydraulic actuators have a higher speed of response with fast starts, stops, and speed reversals.
4. Hydraulic actuators can be operated under continuous, intermittent, reversing, and stalled conditions without damage.
5. Availability of both linear and rotary actuators gives flexibility in design.
6. Because of low leakages in hydraulic actuators, speed drop when loads are applied is small.

On the other hand, several disadvantages tend to limit their use.

1. Hydraulic power is not readily available compared to electric power.
2. Cost of a hydraulic system may be higher than that of a comparable electrical system performing a similar function.
3. Fire and explosion hazards exist unless fire-resistant fluids are used.
4. Because it is difficult to maintain a hydraulic system that is free from leaks, the system tends to be messy.
5. Contaminated oil may cause failure in proper functioning of hydraulic system.
6. As a result of the nonlinear and other complex characteristics involved, the design of sophisticated hydraulic systems is quite involved.
7. Hydraulic circuits have generally poor damping characteristics. If a hydraulic circuit is not designed properly, some unstable phenomena may occur or disappear, depending on the operating condition.

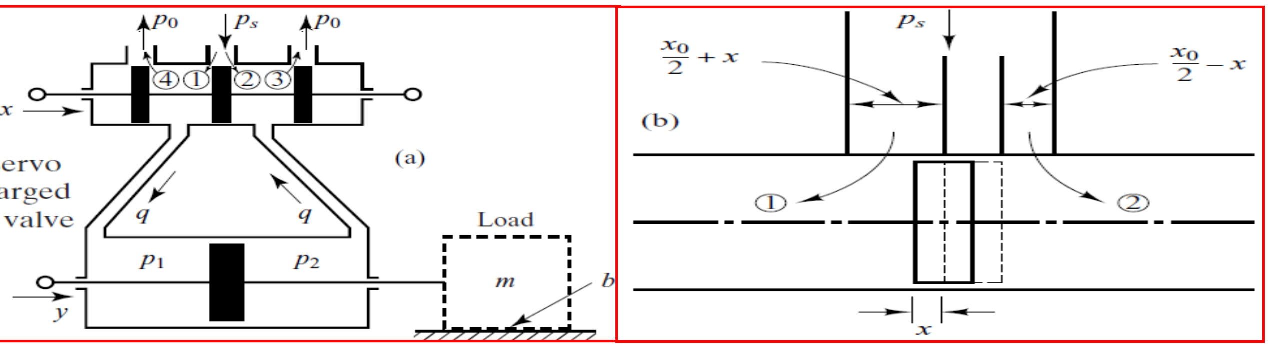
Comments. Particular attention is necessary to ensure that the hydraulic system is stable and satisfactory under all operating conditions. Since the viscosity of hydraulic fluid can greatly affect damping and friction effects of the hydraulic circuits, stability tests must be carried out at the highest possible operating temperature.

Note that most hydraulic systems are nonlinear. Sometimes, however, it is possible to linearize nonlinear systems so as to reduce their complexity and permit solutions that are sufficiently accurate for most purposes. A useful linearization technique for dealing with nonlinear systems was presented in Section 2–7.

Hydraulic Servo System. Figure 4–17(a) shows a hydraulic servomotor. It is essentially a pilot-valve-controlled hydraulic power amplifier and actuator. The pilot valve is a balanced valve, in the sense that the pressure forces acting on it are all balanced. A very large power output can be controlled by a pilot valve, which can be positioned with very little power.

In practice, the ports shown in Figure 4–17(a) are often made wider than the corresponding valves. In such a case, there is always leakage through the valves. Such leak-

Figure 4–17
(a) Hydraulic servo system; (b) enlarged diagram of the valve orifice area.



age improves both the sensitivity and the linearity of the hydraulic servomotor. In the following analysis we shall make the assumption that the ports are made wider than the valves—that is, the valves are underlapped. [Note that sometimes a dither signal, a high-frequency signal of very small amplitude (with respect to the maximum displacement of the valve), is superimposed on the motion of the pilot valve. This also improves the sensitivity and linearity. In this case also there is leakage through the valve.]

We shall apply the linearization technique presented in Section 2–7 to obtain a linearized mathematical model of the hydraulic servomotor. We assume that the valve is underlapped and symmetrical and admits hydraulic fluid under high pressure into a power cylinder that contains a large piston, so that a large hydraulic force is established to move a load.

In Figure 4–17(b) we have an enlarged diagram of the valve orifice area. Let us define the valve orifice areas of ports 1, 2, 3, 4 as A_1, A_2, A_3, A_4, respectively. Also, define the flow rates through ports 1, 2, 3, 4 as q_1, q_2, q_3, q_4, respectively. Note that, since the valve is symmetrical, $A_1 = A_3$ and $A_2 = A_4$. Assuming the displacement x to be small, we obtain $A_1 = A_3 = k\left(\frac{x_0}{2} + x\right)$ $\quad A_2 = A_4 = k\left(\frac{x_0}{2} - x\right)$ where k is a constant.

Furthermore, we shall assume that the return pressure p_o in the return line is small and thus can be neglected. Then, referring to Figure 4–17(a), flow rates through valve orifices are

$$q_1 = c_1 A_1 \sqrt{\frac{2g}{\gamma}(p_s - p_1)} = C_1 \sqrt{p_s - p_1}\left(\frac{x_0}{2} + x\right)$$

$$q_2 = c_2 A_2 \sqrt{\frac{2g}{\gamma}(p_s - p_2)} = C_2 \sqrt{p_s - p_2}\left(\frac{x_0}{2} - x\right)$$

$$q_3 = c_1 A_3 \sqrt{\frac{2g}{\gamma}(p_2 - p_0)} = C_1 \sqrt{p_2 - p_0}\left(\frac{x_0}{2} + x\right) = C_1 \sqrt{p_2}\left(\frac{x_0}{2} + x\right)$$

$$q_4 = c_2 A_4 \sqrt{\frac{2g}{\gamma}(p_1 - p_0)} = C_2 \sqrt{p_1 - p_0}\left(\frac{x_0}{2} - x\right) = C_2 \sqrt{p_1}\left(\frac{x_0}{2} - x\right)$$

where $C_1 = c_1 k\sqrt{2g/\gamma}$ and $C_2 = c_2 k\sqrt{2g/\gamma}$, and γ is the specific weight and is given by $\gamma = \rho g$, where ρ is mass density and g is the acceleration of gravity. The flow rate q to the left-hand side of the power piston is

$$q = q_1 - q_4 = C_1 \sqrt{p_s - p_1}\left(\frac{x_0}{2} + x\right) - C_2 \sqrt{p_1}\left(\frac{x_0}{2} - x\right) \qquad (4\text{–}25)$$

The flow rate from the right-hand side of the power piston to the drain is the same as this q and is given by $q = q_3 - q_2 = C_1\sqrt{p_2}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{p_s - p_2}\left(\frac{x_0}{2} - x\right)$

In the present analysis we assume that the fluid is incompressible. Since the valve is symmetrical, we have $q_1 = q_3$ and $q_2 = q_4$. By equating q_1 and q_3, we obtain

$$p_s - p_1 = p_2 \quad \text{or} \quad p_s = p_1 + p_2$$

If we define the pressure difference across the power piston as Δp or $\Delta p = p_1 - p_2$

then $$p_1 = \frac{p_s + \Delta p}{2}, \qquad p_2 = \frac{p_s - \Delta p}{2}$$

For the symmetrical valve shown in Figure 4–17(a), the pressure in each side of the power piston is $(1/2)p_s$ when no load is applied, or $\Delta p = 0$. As the spool valve is displaced, the pressure in one line increases as the pressure in the other line decreases by the same amount.

In terms of p_s and Δp, we can rewrite the flow rate q given by Equation (4–25) as

$$q = q_1 - q_4 = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + \Delta p}{2}}\left(\frac{x_0}{2} - x\right)$$

Noting that the supply pressure p_s is constant. the flow rate q can be written as a function of the valve displacement x and pressure difference Δp, or

$$q = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + \Delta p}{2}}\left(\frac{x_0}{2} - x\right) = f(x, \Delta p)$$

By applying the linearization technique presented in Section 3–10 to this case, the linearized equation about point $x = \bar{x}, \Delta p = \Delta\bar{p}, q = \bar{q}$ is

$$q - \bar{q} = a(x - \bar{x}) + b(\Delta p - \Delta\bar{p}) \quad \text{where} \quad \bar{q} = f(\bar{x}, \Delta\bar{p}) \qquad (4\text{–}26)$$

$$a = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\,\Delta p=\Delta p} = C_1\sqrt{\frac{p_s - \Delta\bar{p}}{2}} + C_2\sqrt{\frac{p_s + \Delta\bar{p}}{2}}$$

$$b = \left.\frac{\partial f}{\partial \Delta p}\right|_{x=\bar{x},\,\Delta p=\Delta p} = -\left[\frac{C_1}{2\sqrt{2}\sqrt{p_s - \Delta\bar{p}}}\left(\frac{x_0}{2} + \bar{x}\right) + \frac{C_2}{2\sqrt{2}\sqrt{p_s + \Delta\bar{p}}}\left(\frac{x_0}{2} - \bar{x}\right)\right] < 0$$

Coefficients a and b here are called *valve coefficients*. Equation (4–26) is a linearized mathematical model of the spool valve near an operating point $x = \bar{x}, \Delta p = \Delta\bar{p}, q = \bar{q}$. The values of valve coefficients a and b vary with the operating point. Note that $\partial f/\partial \Delta p$ is negative and so b is negative.

Since the normal operating point is the point where $\bar{x} = 0, \Delta\bar{p} = 0, \bar{q} = 0$, near the normal operating point Equation (4–26) becomes

$$q = K_1 x - K_2 \Delta p \qquad (4\text{–}27)$$

where $K_1 = (C_1 + C_2)\sqrt{\frac{p_s}{2}} > 0 \qquad K_2 = (C_1 + C_2)\frac{x_0}{4\sqrt{2}\sqrt{p_s}} > 0$

Equation (4–27) is a linearized mathematical model of the spool valve near the origin ($\bar{x} = 0, \Delta\bar{p} = 0, \bar{q} = 0$.) Note that the region near the origin is most important in this kind of system, because the system operation usually occurs near this point.

Figure 4–18 shows this linearized relationship among q, x, and ΔP. The straight lines shown are the characteristic curves of the linearized hydraulic servomotor. This family of curves consists of equidistant parallel straight lines, parametrized by x.

In the present analysis we assume that the load reactive forces are small, so that the leakage flow rate and oil compressibility can be ignored.

Referring to Figure 4–17(a), we see that the rate of flow of oil q times dt is equal to the power-piston displacement dy times the piston area A times the density of oil ρ. Thus, we obtain

$$A\rho\, dy = q\, dt$$

Notice that for a given flow rate q the larger the piston area A is, the lower will be the velocity dy/dt. Hence, if the piston area A is made smaller, the other variables remaining constant, the velocity dy/dt will become higher. Also, an increased flow rate q will cause an increased velocity of the power piston and will make the response time shorter. Equation (4–27) can now be written as $\Delta P = \frac{1}{K_2}\left(K_1 x - A\rho\frac{dy}{dt}\right)$

The force developed by the power piston is equal to the pressure difference ΔP times the piston area A or Force developed by the power piston $= A\,\Delta P = \frac{A}{K_2}\left(K_1 x - A\rho\frac{dy}{dt}\right)$

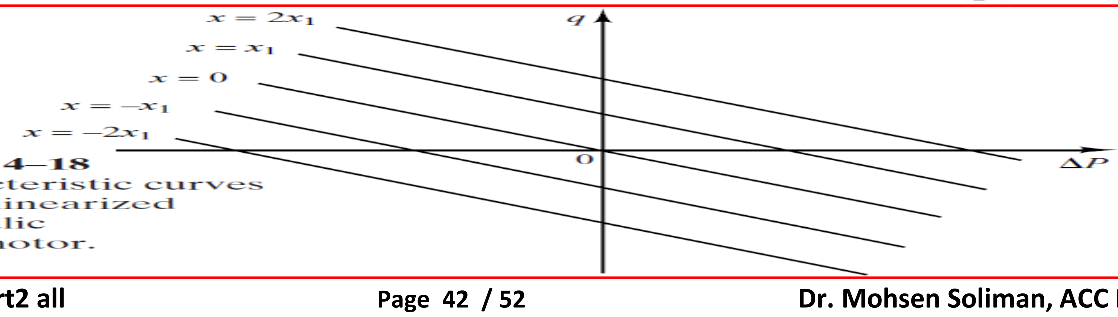


Figure 4–18
Characteristic curves of the linearized hydraulic servomotor.

For a given maximum force, if the pressure difference is sufficiently high, the piston area, or the volume of oil in the cylinder, can be made small. Consequently, to minimize the weight of the controller, we must make the supply pressure sufficiently high.

Assume that the power piston moves a load consisting of a mass and viscous friction. Then the force developed by the power piston is applied to the load mass and friction, and we obtain

$$m\ddot{y} + b\dot{y} = \frac{A}{K_2}(K_1 x - A\rho\dot{y}) \quad \text{or} \quad m\ddot{y} + \left(b + \frac{A^2\rho}{K_2}\right)\dot{y} = \frac{AK_1}{K_2}x \qquad (4\text{–}28)$$

where m is the mass of the load and b is the viscous-friction coefficient.

Assuming that the pilot-valve displacement x is the input and the power-piston displacement y is the output, we find that the transfer function for the hydraulic servomotor is, from Equation (4–28),

$$\frac{Y(s)}{X(s)} = \frac{1}{s\left[\left(\frac{mK_2}{AK_1}\right)s + \frac{bK_2}{AK_1} + \frac{A\rho}{K_1}\right]} = \frac{K}{s(Ts+1)} \qquad (4\text{–}29)$$

where $K = \dfrac{1}{bK_2/AK_1 + A\rho/K_1}$ and $T = \dfrac{mK_2}{bK_2 + A^2\rho}$

From Equation (4–29) we see that this transfer function is of the second order. If the ratio $mK_2/(bK_2 + A^2\rho)$ is negligibly small or the time constant T is negligible, the transfer function $Y(s)/X(s)$ can be simplified to give $\dfrac{Y(s)}{X(s)} = \dfrac{K}{s}$

It is noted that a more detailed analysis shows that if oil leakage, compressibility (including the effects of dissolved air), expansion of pipelines, and the like are taken into consideration, the transfer function becomes $\dfrac{Y(s)}{X(s)} = \dfrac{K}{s(T_1 s + 1)(T_2 s + 1)}$

where T_1 and T_2 are time constants. As a matter of fact, these time constants depend on the volume of oil in the operating circuit. The smaller the volume, the smaller the time constants.

Hydraulic Integral Controller. The hydraulic servomotor shown in Figure 4–19 is a pilot-valve-controlled hydraulic power amplifier and actuator. Similar to the hydraulic servo system shown in Figure 4–17, for negligibly small load mass the servomotor shown in Figure 4–19 acts as an integrator or an integral controller. Such a servomotor constitutes the basis of the hydraulic control circuit.

In the hydraulic servomotor shown in Figure 4–19, the pilot valve (a four-way valve) has two lands on the spool. If the width of the land is smaller than the port in the valve sleeve, the valve is said to be *underlapped*. *Overlapped* valves have a land width greater than the port width. A *zero-lapped* valve has a land width that is identical to the port width. (If the pilot valve is a zero-lapped valve, analyses of hydraulic servomotors become simpler.)

In the present analysis, we assume that hydraulic fluid is incompressible and that the inertia force of the power piston and load is negligible compared to the hydraulic force at the power piston. We also assume that the pilot valve is a zero-lapped valve, and the oil flow rate is proportional to the pilot valve displacement.

Operation of this hydraulic servomotor is as follows. If input x moves the pilot valve to the right, port II is uncovered, and so high-pressure oil enters the right-hand side of the power piston. Since port I is connected to the drain port, the oil in the left-hand side of the power piston is returned to the drain. The oil flowing into the power cylinder is at high pressure; the oil flowing out from the power cylinder into the drain is at low pressure. The resulting difference in pressure on both sides of the power piston will cause it to move to the left.

Note that the rate of flow of oil q (kg/sec) times dt (sec) is equal to the power-piston displacement dy (m) times the piston area A (m^2) times the density of oil ρ (kg/m^3). Therefore,

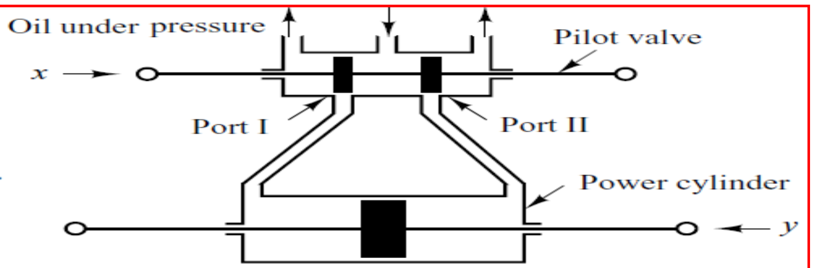
$$A\rho\, dy = q\, dt \qquad (4\text{–}30)$$

Because of the assumption that the oil flow rate q is proportional to the pilot-valve displacement x, we have

$$q = K_1 x \qquad (4\text{–}31)$$

where K_1 is a positive constant. From (4–30) and (4–31) we obtain $A\rho\dfrac{dy}{dt} = K_1 x$

Figure 4–19 Hydraulic servomotor.



The Laplace transform of this last equation, assuming a zero initial condition, gives

$$A\rho s Y(s) = K_1 X(s) \quad \text{or} \quad \frac{Y(s)}{X(s)} = \frac{K_1}{A\rho s} = \frac{K}{s}$$

where $K = K_1/(A\rho)$. Thus the hydraulic servomotor shown in Figure 4–19 acts as an integral controller.

Hydraulic Proportional Controller. It has been shown that the servomotor in Figure 4–19 acts as an integral controller. This servomotor can be modified to a proportional controller by means of a feedback link. Consider the hydraulic controller shown in Figure 4–20(a). The left-hand side of the pilot valve is joined to the left-hand side of the power piston by a link ABC. This link is a floating link rather than one moving about a fixed pivot.

The controller here operates in the following way. If input e moves the pilot valve to the right, port II will be uncovered and high-pressure oil will flow through port II into the right-hand side of the power piston and force this piston to the left. The power piston, in moving to the left, will carry the feedback link ABC with it, thereby moving the pilot valve to the left. This action continues until the pilot piston again covers ports I and II. A block diagram of the system can be drawn as in Figure 4–20(b). The transfer function between $Y(s)$ and $E(s)$ is given by

$$\frac{Y(s)}{E(s)} = \frac{K\, b/(a+b)\, s}{1 + K\, a\, /s(a+b)}$$

Noting that under the normal operating conditions we have $|Ka/[s(a+b)]| \gg 1$, this last equation can be simplified to

$$\frac{Y(s)}{E(s)} = \frac{b}{a} = K_p$$

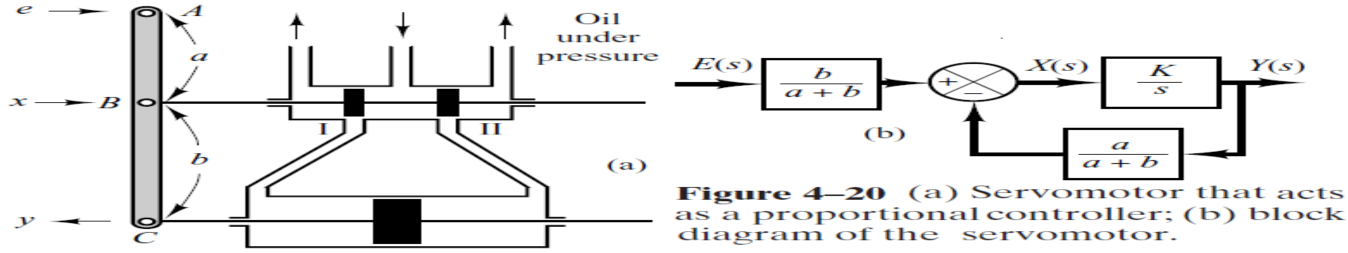


Figure 4–20 (a) Servomotor that acts as a proportional controller; (b) block diagram of the servomotor.

The transfer function between y and e becomes a constant. Thus, the hydraulic controller shown in Figure 4–20(a) acts as a proportional controller, the gain of which is K_p. This gain can be adjusted by effectively changing the lever ratio b/a. (The adjusting mechanism is not shown in the diagram.)

We have thus seen that the addition of a feedback link will cause the hydraulic servomotor to act as a proportional controller.

Dashpots. The dashpot (also called a damper) shown in Figure 4–21(a) acts as a differentiating element. Suppose that we introduce a step displacement to the piston position y. Then the displacement z becomes equal to y momentarily. Because of the spring force, however, the oil will flow through the resistance R and the cylinder will come back to the original position. The curves y versus t and z versus t are shown in Figure 4–21(b).

Let us derive the transfer function between the displacement z and displacement y. Define the pressures existing on the right and left sides of the piston as $P_1(\text{lb}_f/\text{in.}^2)$ and $P_2(\text{lb}_f/\text{in.}^2)$, respectively. Suppose that the inertia force involved is negligible. Then the force acting on the piston must balance the spring force. Thus

$A(P_1 - P_2) = kz$ where A = piston area, in.^{2} k = spring constant, lb$_f$/in.

The flow rate q is given by $q = \dfrac{P_1 - P_2}{R}$ where q = flow rate through the restriction, lb/sec R = resistance to flow at the restriction, lb$_f$-sec/in.^{2}-lb

Since the flow through the restriction during dt seconds must equal the change in the mass of oil to the left of the piston during the same dt seconds, we obtain

$$q\,dt = A\rho(dy - dz)$$

where ρ = density, lb/in.^{3}. (We assume that the fluid is incompressible or ρ = constant.) This last equation can be rewritten as $\dfrac{dy}{dt} - \dfrac{dz}{dt} = \dfrac{q}{A\rho} = \dfrac{P_1 - P_2}{RA\rho} = \dfrac{kz}{RA^2\rho}$

Figure 4–21
(a) Dashpot; (b) step change in y and the corresponding change in z plotted versus t; (c) block diagram of the dashpot.

or $$\frac{dy}{dt} = \frac{dz}{dt} + \frac{kz}{RA^2\rho}$$

Taking the Laplace transforms of both sides of this last equation, assuming zero initial conditions, we obtain $sY(s) = sZ(s) + \dfrac{k}{RA^2\rho}Z(s)$

The transfer function of this system thus becomes $\dfrac{Z(s)}{Y(s)} = \dfrac{s}{s + k/RA^2\rho}$

Let us define $RA^2\rho/k = T$. (Note that $RA^2\rho/k$ has the dimension of time.) Then

$$\frac{Z(s)}{Y(s)} = \frac{Ts}{Ts + 1} = \frac{1}{1 + 1/Ts}$$

Clearly, the dashpot is a differentiating element. Figure 4–21(c) shows a block diagram representation for this system.

Obtaining Hydraulic Proportional-Plus-Integral Control Action. Figure 4–22(a) shows a schematic diagram of a hydraulic proportional-plus-integral controller. A block diagram of this controller is shown in Figure 4–22(b). The transfer function $Y(s)/E(s)$ is given by

$$\frac{Y(s)}{E(s)} = \frac{bK/s(a+b)}{1 + [Ka/(a+b)][T/(Ts+1)]}$$

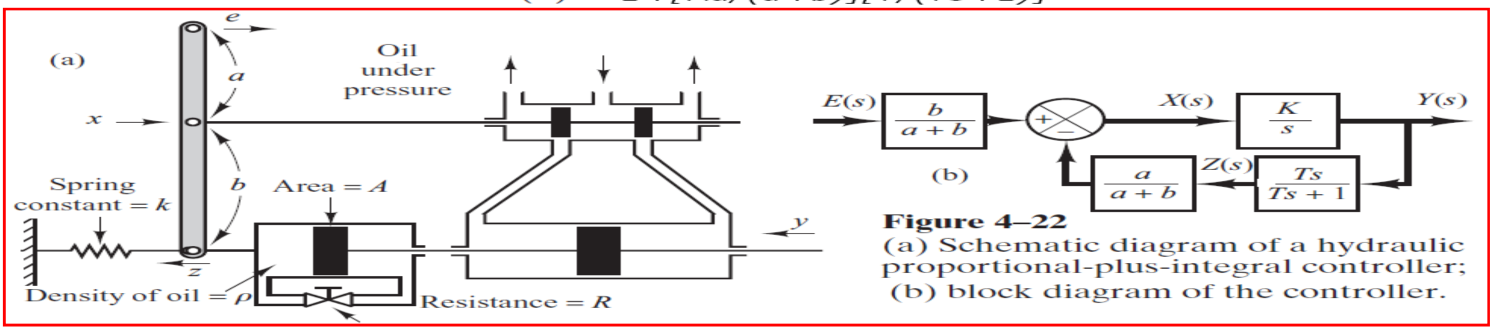


Figure 4–22
(a) Schematic diagram of a hydraulic proportional-plus-integral controller; (b) block diagram of the controller.

In such a controller, under normal operation $|KaT/[(a+b)(Ts+1)]| \gg 1$, with the result that $\dfrac{Y(s)}{E(s)} = K_p\left(1 + \dfrac{1}{T_i s}\right)$ where $K_p = \dfrac{b}{a}$, $T_i = T = \dfrac{RA^2\rho}{k}$

Thus the controller shown in Figure 4–22(a) is a proportional-plus-integral controller (PI controller).

Obtaining Hydraulic Proportional-Plus-Derivative Control Action. Figure 4–23(a) shows a schematic diagram of a hydraulic proportional-plus-derivative controller. The cylinders are fixed in space and the pistons can move. For this system, notice that

$$k(y - z) = A(P_2 - P_1) \qquad q = \frac{P_2 - P_1}{R} \qquad q\,dt = \rho A\,dz$$

Hence $y = z + \dfrac{A}{k}qR = z + \dfrac{RA^2\rho}{k}\dfrac{dz}{dt}$ or $\dfrac{Z(s)}{Y(s)} = \dfrac{1}{Ts + 1}$

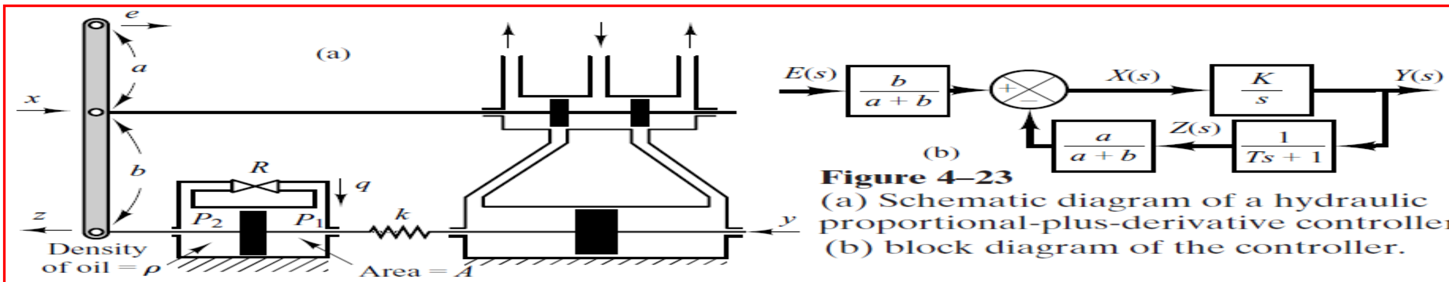


Figure 4–23
(a) Schematic diagram of a hydraulic proportional-plus-derivative controller; (b) block diagram of the controller.

where $$T = \frac{RA^2\rho}{k}$$

A block diagram for this system is shown in Figure 4–23(b). From the block diagram the transfer function $Y(s)/E(s)$ can be obtained as

$$\frac{Y(s)}{E(s)} = \frac{bK/s(a+b)}{1+[a/(a+b)(K/s)(1/Ts+1)]}$$

Under normal operation we have $|aK/[(a+b)s(Ts+1)]| \gg 1$. Hence

$$\frac{Y(s)}{E(s)} = K_p(1+Ts) \qquad \text{where} \quad K_p = \frac{b}{a}, \quad T = \frac{RA^2\rho}{k}$$

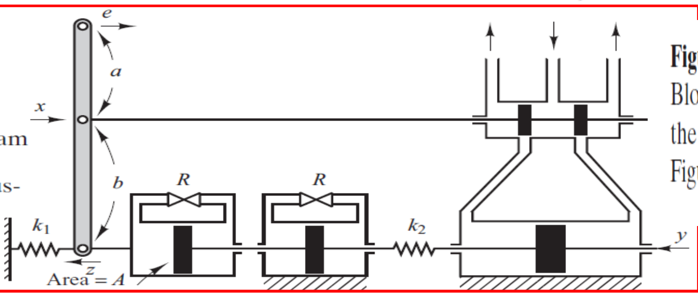
Thus the controller shown in Figure 4–23(a) is a proportional-plus-derivative controller (PD controller).

Obtaining Hydraulic Proportional-Plus-Integral-Plus-Derivative Control Action. Figure 4–24 shows a schematic diagram of a hydraulic proportional-plus-integral-plus-derivative controller. It is a combination of the proportional-plus-integral controller and proportional-plus derivative controller.

If the two dashpots are identical except the piston shafts, the transfer function $Z(s)/Y(s)$ can be obtained as follows: $$\frac{Z(s)}{Y(s)} = \frac{T_1 s}{T_1T_2s^2 + (T_1 + 2T_2)s + 1}$$

(For the derivation of this transfer function, refer to Problem **A–4–9**.)

Figure 4–24 Schematic diagram of a hydraulic proportional-plus-integral-plus-derivative controller.



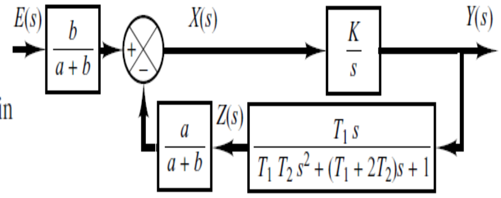


Figure 4–25 Block diagram for the system shown in Figure 4–24.

A block diagram for this system is shown in Figure 4–25. The transfer function $Y(s)/E(s)$ can be obtained as

$$\frac{Y(s)}{E(s)} = \frac{b}{a+b}\,\frac{K/s}{1+[a/(a+b)][K/s]\{T_1 s/T_1T_2s^2+(T_1+2T_2)s+1\}}$$

Under normal circumstances we design the system such that

$$\left|\frac{a}{a+b}\,\frac{K}{s}\,\frac{T_1 s}{T_1T_2s^2+(T_1+2T_2)s+1}\right| \gg 1 \quad \text{then} \quad \frac{Y(s)}{E(s)} = \frac{b}{a}\,\frac{T_1T_2s^2+(T_1+2T_2)s+1}{T_1 s}$$

$$\frac{Y(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s \qquad \text{where} \quad K_p = \frac{b}{a}\,\frac{T_1+2T_2}{T_1}, \quad K_i = \frac{b}{a}\,\frac{1}{T_1}, \quad K_d = \frac{b}{a}T_2$$

Thus, the controller shown in Figure 4–24 is a proportional-plus-integral-plus-derivative controller (PID controller).

4–5 THERMAL SYSTEMS

Thermal systems are those that involve the transfer of heat from one substance to another. Thermal systems may be analyzed in terms of resistance and capacitance, although the thermal capacitance and thermal resistance may not be represented accurately as lumped parameters, since they are usually distributed throughout the substance. For precise analysis, distributed-parameter models must be used. Here, however, to simplify the analysis we shall assume that a thermal system can be represented by a lumped-parameter model, that substances that are characterized by resistance to heat flow have negligible heat capacitance, and that substances that are characterized by heat capacitance have negligible resistance to heat flow.

There are three different ways heat can flow from one substance to another: conduction, convection, and radiation. Here we consider only conduction and convection. (Radiation heat transfer is appreciable only if the temperature of the emitter is very high compared to that of the receiver. Most thermal processes in process control systems do not involve radiation heat transfer.)

For conduction or convection heat transfer, $q = K\,\Delta\theta$ where
q = heat flow rate, kcal/sec $\quad \Delta\theta$ = temp difference, °C $\quad K$ = coefficient, kcal/sec°C
The coefficient K is given by $K = \frac{kA}{\Delta X}$, for conduction $\quad K = HA$, for convection
where k = thermal conductivity, kcal/m sec °C $\quad A$ = area normal to heat flow, m^2
ΔX = thickness of conductor, m $\quad H$ = convection coefficient, kcal/m^2 sec °C

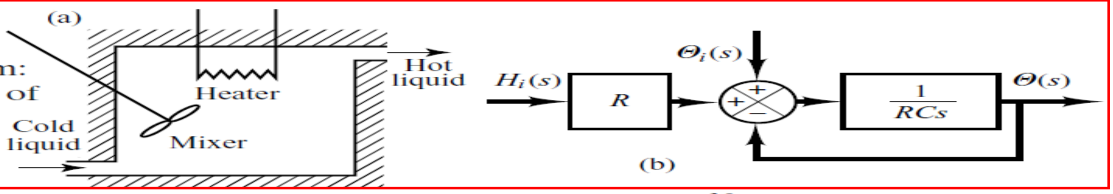
Thermal System. Consider the system shown in Figure 4–26(a). It is assumed that the tank is insulated to eliminate heat loss to the surrounding air. It is also assumed that there is no heat storage in the insulation and that the liquid in the tank is perfectly mixed so that it is at a uniform temperature. Thus, a single temperature is used to describe the temperature of the liquid in the tank and of the outflowing liquid.

Let us define $\bar{\Theta}_i$ = steady-state temperature of inflowing liquid, °C
$\bar{\Theta}_o$ = steady-state temp of outflowing liquid, °C $\quad R$ = thermal resistance, °C sec/kcal
G = steady-state liquid flow rate, kg/sec $\quad M$ = mass of liquid in tank, kg
c = specific heat of liquid, kcal/kg °C $\quad C$ = thermal capacitance, kcal/°C
$\bar{H}$ = steady-state heat input rate, kcal/sec

Assume that the temperature of the inflowing liquid is kept constant and that the heat input rate to the system (heat supplied by the heater) is suddenly changed from $\bar{H}$ to $\bar{H} + h_i$, where h_i represents a small change in the heat input rate. The heat outflow rate will then change gradually from $\bar{H}$ to $\bar{H} + h_o$. The temperature of the outflowing liquid will also be changed from $\bar{\Theta}_o$ to $\bar{\Theta}_o + \theta$. For this case, h_o, C, and R are obtained, respectively, as $h_o = Gc\theta$ $\quad C = Mc \quad$ $R = \dfrac{\theta}{h_o} = \dfrac{1}{Gc}$

The heat-balance equation for this system is $C\,d\theta = (h_i - h_o)\,dt$

Figure 4–26
(a) Thermal system:
(b) block diagram of
the system.



or $C\dfrac{d\theta}{dt} = h_i - h_o$ which may be rewritten as $RC\dfrac{d\theta}{dt} + \theta = Rh_i$

Note that the time constant of the system is equal to RC or M/G seconds. The transfer function relating θ and h_i is given by

$\dfrac{\Theta(s)}{H_i(s)} = \dfrac{R}{RCs + 1}$ where $\Theta(s) = \mathscr{L}[\theta(t)]$ and $H_i(s) = \mathscr{L}[h_i(t)]$.

In practice, the temperature of the inflowing liquid may fluctuate and may act as a load disturbance. (If a constant outflow temperature is desired, an automatic controller may be installed to adjust the heat inflow rate to compensate for the fluctuations in the temperature of the inflowing liquid.) If the temperature of the inflowing liquid is suddenly changed from $\bar{\Theta}_i$ to $\bar{\Theta}_i + \theta_i$ while the heat input rate H and the liquid flow rate G are kept constant, then the heat outflow rate will be changed from $\bar{H}$ to $\bar{H} + h_o$, and the temperature of the outflowing liquid will be changed from $\bar{\Theta}_o$ to $\bar{\Theta}_o + \theta$. The heat-balance equation for this case is

$C\,d\theta = (Gc\theta_i - h_o)\,dt$ or $C\dfrac{d\theta}{dt} = Gc\theta_i - h_o$ which may be rewritten $RC\dfrac{d\theta}{dt} + \theta = \theta_i$

The transfer function relating θ and θ_i is given by

$\dfrac{\Theta(s)}{\Theta_i(s)} = \dfrac{1}{RCs + 1}$ where $\Theta(s) = \mathscr{L}[\theta(t)]$ and $\Theta_i(s) = \mathscr{L}[\theta_i(t)]$.

If the present thermal system is subjected to changes in both the temperature of the inflowing liquid and the heat input rate, while the liquid flow rate is kept constant, the change θ in the temperature of the outflowing liquid can be given by the following equation: $RC\dfrac{d\theta}{dt} + \theta = \theta_i + Rh_i$

A block diagram corresponding to this case is shown in Figure 4–26(b). Notice that the system involves two inputs.

EXAMPLE PROBLEMS AND SOLUTIONS

A–4–1.
In the liquid-level system of Figure 4–27 assume that the outflow rate Q m^3/sec through the outflow valve is related to the head H m by $Q = K\sqrt{H} = 0.01\sqrt{H}$

Assume also that when the inflow rate Q_i is 0.015 m^3/sec the head stays constant. For $t < 0$ the system is at steady state ($Q_i = 0.015$ m^3/sec). At $t = 0$ the inflow valve is closed and so there is no inflow for $t \geq 0$. Find the time necessary to empty the tank to half the original head. The capacitance C of the tank is 2 m^2.

Solution. When the head is stationary, the inflow rate equals the outflow rate. Thus head H_o at $t = 0$ is obtained from $0.015 = 0.01\sqrt{H_o}$ or $H_o = 2.25$ m

The equation for the system for $t > 0$ is $-C\,dH = Q\,dt$ or $\dfrac{dH}{dt} = -\dfrac{Q}{C} = \dfrac{-0.01\sqrt{H}}{2}$

Hence $\dfrac{dH}{\sqrt{H}} = -0.005\,dt$

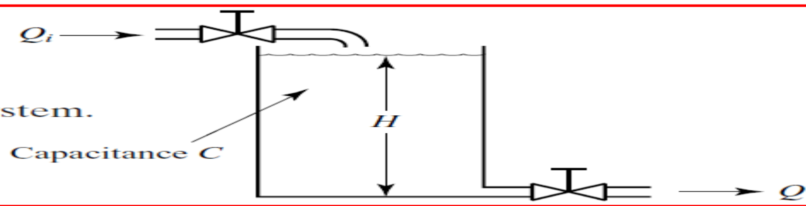
Assume that, at $t = t_1$, $H = 1.125$ m. Integrating both sides of this last equation, we obtain

$\displaystyle\int_{2.25}^{1.125}\frac{dH}{\sqrt{H}} = \int_0^{t_1}(-0.005)\,dt = -0.005t_1$ It follows that $2\sqrt{H}\Big|_{2.25}^{1.125} = 2\sqrt{1.125} - 2\sqrt{2.25}$

$= -0.005t_1$ or $t_1 = 175.7$

Thus, the head becomes half the original value (2.25 m) in 175.7 sec.

Figure 4–27
Liquid-level system.



A–4–2.
Consider the liquid-level system shown in Figure 4–28. In the system, $\bar{Q}_1$ and $\bar{Q}_2$ are steady-state inflow rates and $\bar{H}_1$ and $\bar{H}_2$ are steady-state heads. The quantities $q_{i1}, q_{i2}, h_1, h_2, q_1$, and q_o are considered small. Obtain a state-space representation for the system when h_1 and h_2 are the outputs and q_{i1} and q_{i2} are the inputs.

Solution. The equations for the system are $C_1\,dh_1 = (q_{i1} - q_1)\,dt$ (4–32)

$\dfrac{h_1 - h_2}{R_1} = q_1$ (4–33) $\quad C_2\,dh_2 = (q_1 + q_{i2} - q_o)\,dt$ (4–34) $\quad \dfrac{h_2}{R_2} = q_o$ (4–35)

Elimination of q_1 from Equation (4–32) using Equation (4–33) results in

$$\frac{dh_1}{dt} = \frac{1}{C_1}\left(q_{i1} - \frac{h_1 - h_2}{R_1}\right) \qquad (4\text{–}36)$$

Eliminating q_1 and q_o from Equation (4–34) by using Equations (4–33) and (4–35) gives

$\dfrac{dh_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{h_1 - h_2}{R_1} + q_{i2} - \dfrac{h_2}{R_2}\right)$ (4–37) Define state variables x_1 and x_2 by $x_1 = h_1 \quad x_2 = h_2$ the input variables u_1 and u_2 by $u_1 = q_{i1} \quad u_2 = q_{i2}$ and the output variables y_1 and y_2 by $y_1 = h_1 = x_1 \quad y_2 = h_2 = x_2$

Then Equations (4–36) and (4–37) can be written as

$\dot{x}_1 = -\dfrac{1}{R_1C_1}x_1 + \dfrac{1}{R_1C_1}x_2 + \dfrac{1}{C_1}u_1 \qquad \dot{x}_2 = \dfrac{1}{R_1C_2}x_1 - \left(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2}\right)x_2 + \dfrac{1}{C_2}u_2$

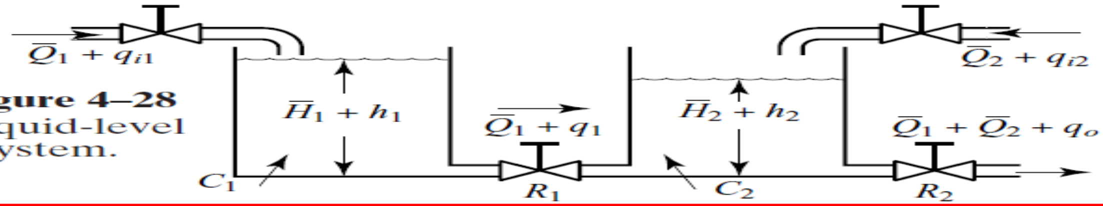


Figure 4–28 Liquid-level system.

In the form of the standard vector-matrix representation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1C_1} & \dfrac{1}{R_1C_1} \\ \dfrac{1}{R_1C_2} & -\left(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C_1} & 0 \\ 0 & \dfrac{1}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

which is the state equation, and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is the output equation.

A–4–3.

The value of the gas constant for any gas may be determined from accurate experimental observations of simultaneous values of p, v, and T.

Obtain the gas constant R_{air} for air. Note that at 32°F and 14.7 psia the specific volume of air is 12.39 ft^3/lb. Then obtain the capacitance of a 20-ft^3 pressure vessel that contains air at 160°F. Assume that the expansion process is isothermal.

Solution.

$$R_{air} = \frac{pv}{T} = \frac{14.7 \times 144 \times 12.39}{460 + 32} = 53.3 \text{ ft-lb}_f/\text{lb}°\text{R}$$

Referring to Equation (4–12), the capacitance of a 20-ft^3 pressure vessel is

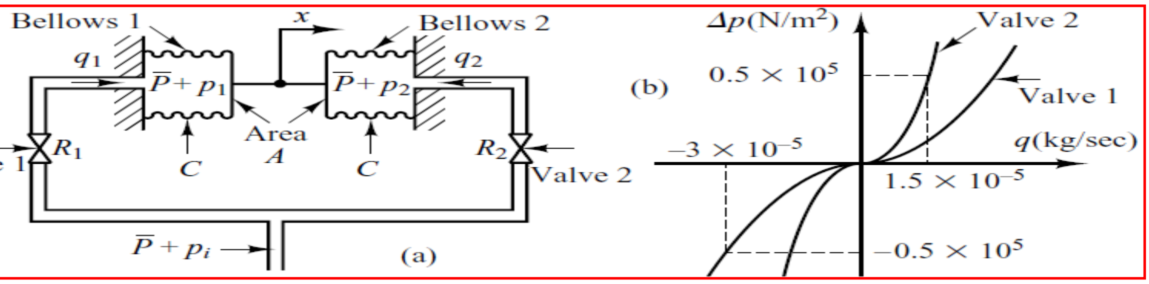
$$C = \frac{V}{nR_{air}T} = \frac{20}{1 \times 53.3 \times 620} = 6.05 \times 10^{-4} \frac{\text{lb}}{\text{lb}_f/\text{ft}^2}$$

Note that in terms of SI units, R_{air} is given by $\qquad R_{air} = 287$ N-m/kg K

A–4–4.

In the pneumatic pressure system of Figure 4–29(a), assume that, for $t < 0$, the system is at steady state and that the pressure of the entire system is $\bar{P}$. Also, assume that the two bellows are identical. At $t = 0$, the input pressure is changed from $\bar{P}$ to $\bar{P} + p_i$. Then the pressures in bellows 1 and 2 will change from $\bar{P}$ to $\bar{P} + p_1$ and from $\bar{P}$ to $\bar{P} + p_2$, respectively. The capacity (volume) of each bellows is 5×10^{-4} m^3, and the operating-pressure difference Δp (difference between p_i and p_1 or difference between p_i and p_2) is between -0.5×10^5 N/m^2 and 0.5×10^5 N/m^2. The corresponding mass flow rates (kg/sec) through the valves are shown in Figure 4–29(b). Assume that the bellows expand or contract linearly with the air pressures applied to them, that the equivalent spring constant of the bellows system is $k = 1 \times 10^5$ N/m, and that each bellows has area $A = 15 \times 10^{-4}$ m^2.

Figure 4–29 (a) Pneumatic pressure system; (b) pressure-difference-versus-mass-flow-rate curves.



Defining the displacement of the midpoint of the rod that connects two bellows as x, find the transfer function $X(s)/P_i(s)$. Assume that the expansion process is isothermal and that the temperature of the entire system stays at 30°C. Assume also that the polytropic exponent n is 1.

Solution. Referring to Section 4–3, transfer function $P_1(s)/P_i(s)$ can be obtained as

$$\frac{P_1(s)}{P_i(s)} = \frac{1}{R_1Cs + 1} \qquad (4–38)$$

Similarly, transfer function $P_2(s)/P_i(s)$ is

$$\frac{P_2(s)}{P_i(s)} = \frac{1}{R_2Cs + 1} \qquad (4–39)$$

The force acting on bellows 1 in the x direction is $A(\bar{P} + p_1)$, and the force acting on bellows 2 in the negative x direction is $A(\bar{P} + p_2)$. The resultant force balances with kx, the equivalent spring force of the corrugated sides of the bellows.

$$A(p_1 - p_2) = kx \qquad \text{or} \qquad A[P_1(s) - P_2(s)] = kX(s) \qquad (4–40)$$

Referring to Equations (4–38) and (4–39), we see that

$$P_1(s) - P_2(s) = \left(\frac{1}{R_1Cs + 1} - \frac{1}{R_2Cs + 1}\right)P_i(s) = \frac{R_2Cs - R_1Cs}{(R_1Cs + 1)(R_2Cs + 1)}P_i(s)$$

By substituting this last equation into Equation (4–40) and rewriting, the transfer function $X(s)/P_i(s)$ is obtained as

$$\frac{X(s)}{P_i(s)} = \frac{A}{k}\frac{(R_2C - R_1C)s}{(R_1Cs + 1)(R_2Cs + 1)} \qquad (4–41)$$

The numerical values of average resistances R_1 and R_2 are

$$R_1 = \frac{d\,\Delta p}{dq_1} = \frac{0.5 \times 10^5}{3 \times 10^{-5}} = 0.167 \times 10^{10} \frac{\text{N/m}^2}{\text{kg/sec}} \qquad R_2 = \frac{d\,\Delta p}{dq_2} = \frac{0.5 \times 10^5}{1.5 \times 10^{-5}} = 0.333 \times 10^{10} \frac{\text{N/m}^2}{\text{kg/sec}}$$

The numerical value of capacitance C of each bellows is

$$C = \frac{V}{nR_{air}T} = \frac{5 \times 10^{-4}}{1 \times 287 \times (273 + 30)} = 5.75 \times 10^{-9} \frac{\text{kg}}{\text{N/m}^2}$$

where $R_{air} = 287$ N-m/kg K. (See Problem **A–4–3**.) Consequently,

$$R_1C = 0.167 \times 10^{10} \times 5.75 \times 10^{-9} = 9.60 \text{ sec} \qquad R_2C = 0.333 \times 10^{10} \times 5.75 \times 10^{-9} = 19.2 \text{ sec}$$

By substituting the numerical values for A, k, R_1C, and R_2C into Equation (4–41), we obtain

$$\frac{X(s)}{P_i(s)} = \frac{1.44 \times 10^{-7}s}{(9.6s + 1)(19.2s + 1)}$$

A–4–5.
Draw a block diagram of the pneumatic controller shown in Figure 4–30. Then derive the transfer function of this controller. Assume that $R_d \ll R_i$. Assume also that the two bellows are identical.

If the resistance R_d is removed (replaced by the line-sized tubing), what control action do we get? If the resistance R_i is removed (replaced by the line-sized tubing), what control action do we get?

Solution. Let us assume that when $e = 0$ the nozzle–flapper distance is equal to $\bar{X}$ and the control pressure is equal to $\bar{P}_c$. In the present analysis, we shall assume small deviations from the respective reference values as follows:

e = small error signal x = small change in flapper distance p_c = small change in control pressure

p_{I} = small pressure change in bellows I due to small change in the control pressure

p_{II} = small pressure change in bellows II due to small change in the control pressure

y = small displacement at the lower end of the flapper

In this controller, p_c is transmitted to bellows I through the resistance R_d. Similarly, p_c is transmitted to bellows II through the series of resistances R_d and R_i. The relationship between p_I and p_c is

$$\frac{P_I(s)}{P_c(s)} = \frac{1}{R_d C s + 1} = \frac{1}{T_d s + 1}$$

where $T_d = R_d C$ = derivative time. Similarly, p_{II} and p_{I} are related by the transfer function

$$\frac{P_{\mathrm{II}}(s)}{P_{\mathrm{I}}(s)} = \frac{1}{R_i C s + 1} = \frac{1}{T_i s + 1} \qquad \text{where } T_i = R_i C = \text{integral time.}$$

The force-balance equation for the two bellows is $(p_{\mathrm{I}} - p_{\mathrm{II}})A = k_s y$

where k_s is the stiffness of the two connected bellows and A is the cross-sectional area of the bellows. The relationship among the variables e, x, and y is $x = \frac{b}{a+b}e - \frac{a}{a+b}y$

The relationship between p_c and x is $p_c = Kx \qquad (K > 0)$

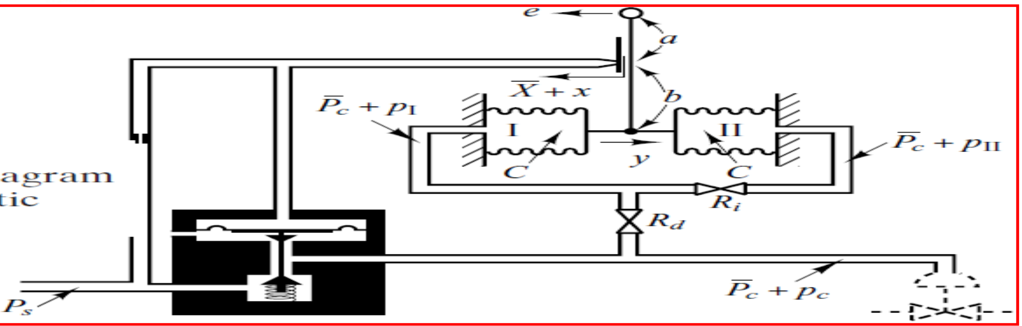


Figure 4–30 Schematic diagram of a pneumatic controller.

From the equations just derived, a block diagram of the controller can be drawn, as shown in Figure 4–31(a). Simplification of this block diagram results in Figure 4–31(b).

The transfer function between $P_c(s)$ and $E(s)$ is

$$\frac{P_c(s)}{E(s)} = \frac{Kb/(a+b)}{1 + KaAT_i s/[(a+b)k_s(T_i s+1)(T_d s+1)]}$$

For a practical controller, under normal operation $|KaAT_i s/[(a+b)k_s(T_i s+1)(T_d s+1)]|$ is very much greater than unity and $T_i \gg T_d$. Therefore, the transfer function can be simplified as follows:

$$\frac{P_c(s)}{E(s)} \doteq \frac{bk_s(T_i s+1)(T_d s+1)}{aAT_i s} = \frac{bk_s}{aA}\left(\frac{T_i + T_d}{T_i} + \frac{1}{T_i s} + T_d s\right)$$

$$\doteq K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \qquad \text{where} \quad K_p = \frac{bk_s}{aA}$$

Thus the controller shown in Figure 4–30 is a proportional-plus-integral-plus-derivative one.

If the resistance R_d is removed, or $R_d = 0$, the action becomes that of a proportional-plus-integral controller.

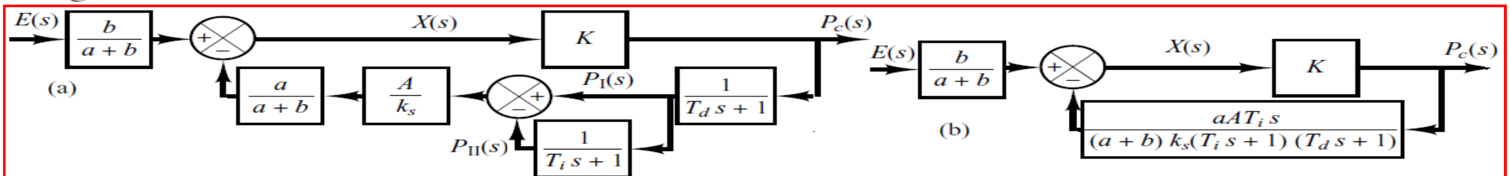
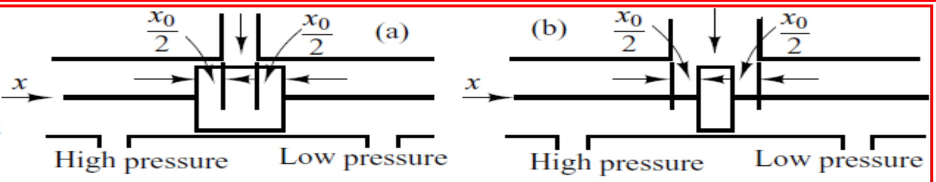


Figure 4–31 (a) Block diagram of the pneumatic controller shown in Figure 4–30; (b) simplified block diagram.

Figure 4–32
(a) Overlapped spool valve; (b) underlapped spool valve.



If the resistance R_i is removed, or $R_i = 0$, the action becomes that of a narrow-band proportional, or two-position, controller. (Note that the actions of two feedback bellows cancel each other, and there is no feedback.)

A–4–6.
Actual spool valves are either overlapped or underlapped because of manufacturing tolerances. Consider the overlapped and underlapped spool valves shown in Figures 4–32(a) and (b). Sketch curves relating the uncovered port area A versus displacement x.

Solution. For the overlapped valve, a dead zone exists between $-\frac{1}{2}x_0$ and $\frac{1}{2}x_0$, or $-\frac{1}{2}x_0 < x < \frac{1}{2}x_0$. The curve for uncovered port area A versus displacement x is shown in Figure 4–33(a). Such an overlapped valve is unfit as a control valve.

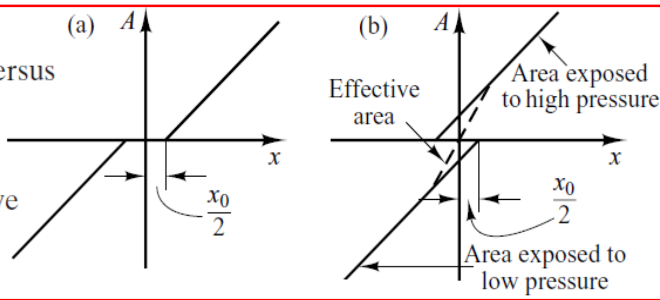
For the underlapped valve, the curve for port area A versus displacement x is shown in Figure 4–33(b). The effective curve for the underlapped region has a higher slope, meaning a higher sensitivity. Valves used for controls are usually underlapped.

A–4–7.
Figure 4–34 shows a hydraulic jet-pipe controller. Hydraulic fluid is ejected from the jet pipe. If the jet pipe is shifted to the right from the neutral position, the power piston moves to the left, and vice versa. The jet-pipe valve is not used as much as the flapper valve because of large null flow, slower response, and rather unpredictable characteristics. Its main advantage lies in its insensitivity to dirty fluids.

Suppose that the power piston is connected to a light load so that the inertia force of the load element is negligible compared to the hydraulic force developed by the power piston. What type of control action does this controller produce?

Solution. Define the displacement of the jet nozzle from the neutral position as x and the displacement of the power piston as y. If the jet nozzle is moved to the right by a small displace-

Figure 4–33
(a) Uncovered-port-area-A-versus displacement-x curve for the overlapped valve;
(b) uncovered-port-area-A-versus-displacement-x curve for the underlapped valve.



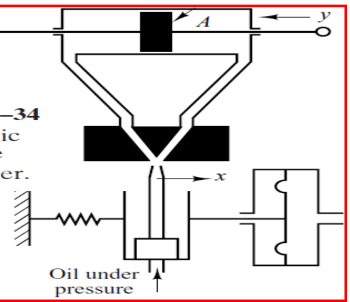


Figure 4–34 Hydraulic jet-pipe controller.

ment x, the oil flows to the right side of the power piston, and the oil in the left side of the power piston is returned to the drain. The oil flowing into the power cylinder is at high pressure; the oil flowing out from the power cylinder into the drain is at low pressure. The resulting pressure difference causes the power piston to move to the left.

For a small jet-nozzle displacement x, the flow rate q to the power cylinder is proportional to x; that is, $q = K_1 x$ For the power cylinder, $A\rho\, dy = q\, dt$ where A is the power-piston area and ρ is the density of oil. Hence $\frac{dy}{dt} = \frac{q}{A\rho} = \frac{K_1}{A\rho} x = Kx$

where $K = K_1/(A\rho)$ = constant. The transfer function $Y(s)/X(s)$ is thus $\frac{Y(s)}{X(s)} = \frac{K}{s}$

The controller produces the integral control action.

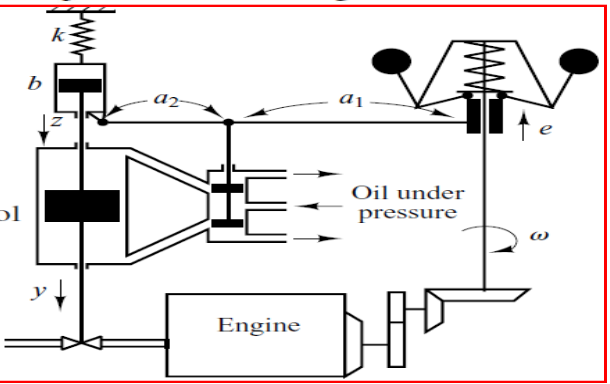
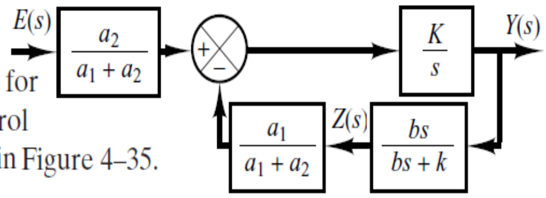


Figure 4–35 Speed control system.

Figure 4–36 Block diagram for the speed control system shown in Figure 4–35.



A–4–8.
Explain the operation of the speed control system shown in Figure 4–35.

Solution. If the engine speed increases, the sleeve of the fly-ball governor moves upward. This movement acts as the input to the hydraulic controller. A positive error signal (upward motion of the sleeve) causes the power piston to move downward, reduces the fuel-valve opening, and decreases the engine speed. A block diagram for the system is shown in Figure 4–36.

From the block diagram the transfer function $Y(s)/E(s)$ can be obtained as

$$\frac{Y(s)}{E(s)} = \frac{a_2}{a_1 + a_2} \frac{K/s}{1+[a_1/(a_1+a_2)][bs/(bs+k)][K/s]}$$

If the following condition applies,

$\left| \frac{a_1}{a_1 + a_2} \frac{bs}{bs + k} \frac{K}{s} \right| \gg 1$ the transfer function $Y(s)/E(s)$ becomes

$$\frac{Y(s)}{E(s)} \doteq \frac{a_2}{a_1 + a_2} \frac{a_1 + a_2}{a_1} \frac{bs + k}{bs} = \frac{a_2}{a_1}\left(1 + \frac{k}{bs}\right)$$

The speed controller has proportional-plus-integral control action.

A–4–9.
Derive the transfer function $Z(s)/Y(s)$ of the hydraulic system shown in Figure 4–37. Assume that the two dashpots in the system are identical ones except the piston shafts.

Solution. In deriving the equations for the system, we assume that force F is applied at the right end of the shaft causing displacement y. (All displacements y, w, and z are measured from respective equilibrium positions when no force is applied at the right end of the shaft.) When force F is applied, pressure P_1 becomes higher than pressure P_1', or $P_1 > P_1'$. Similarly, $P_2 > P_2'$.

For the force balance, we have the following equation:

$k_2(y - w) = A(P_1 - P_1') + A(P_2 - P_2')$ (4–42) Since $k_1 z = A(P_1 - P_1')$ (4–43)

and $q_1 = \frac{P_1 - P_1'}{R}$ we have $k_1 z = ARq_1$ Also, since $q_1\, dt = A(dw - dz)\rho$

we have $q_1 = A(\dot{w} - \dot{z})\rho$ or $\dot{w} - \dot{z} = \frac{k_1 z}{A^2 R\rho}$

Define $A^2 R\rho = B$. (B is the viscous-friction coefficient.) Then $\dot{w} - \dot{z} = \frac{k_1}{B} z$ (4–44)

Also, for right-hand-side dashpot we have $q_2\, dt = A\rho\, dw$ Since $q_2 = (P_2 - P_2')/R$, we obtain

$\dot{w} = \frac{q_2}{A\rho} = \frac{A(P_2 - P_2')}{A^2 R\rho}$ or $A(P_2 - P_2') = B\dot{w}$ (4–45)

Substituting Equations (4–43) and (4–45) into (4–42), we have $k_2 y - k_2 w = k_1 z + B\dot{w}$

Taking the Laplace transform of this last equation, assuming zero initial condition, we obtain

$$k_2 Y(s) = (k_2 + Bs)W(s) + k_1 Z(s) \qquad (4–46)$$

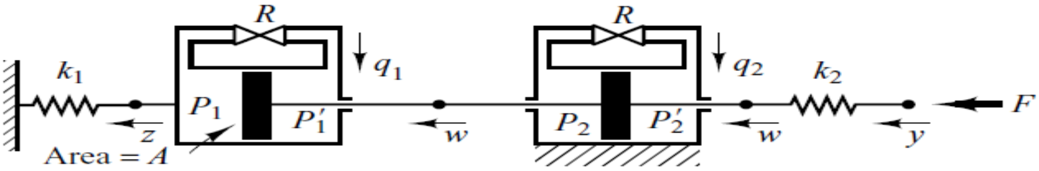


Figure 4–37 Hydraulic system.

Taking the Laplace transform of Equation (4–44), assuming zero initial condition, we obtain

$$W(s) = \frac{k_1 + Bs}{Bs} Z(s) \qquad (4\text{–}47)$$

By using Equation (4–47) to eliminate $W(s)$ from Equation (4–46), we obtain

$k_2 Y(s) = (k_2 + Bs)\frac{k_1 + Bs}{Bs} Z(s) + k_1 Z(s)$ from which we obtain transfer function $Z(s)/Y(s)$ to be

$$\frac{Z(s)}{Y(s)} = \frac{k_2 s}{Bs^2 + (2k_1 + k_2)s + k_1 k_2/B}$$

Multiplying $B/(k_1 k_2)$ to both the numerator and denominator of this last equation, we get

$$\frac{Z(s)}{Y(s)} = \frac{Bs/k_1}{B^2 s^2/k_1 k_2 + (2B/k_2 + B/k_1)s + 1}$$

Define $B/k_1 = T_1$, $B/k_2 = T_2$. Then the transfer function $Z(s)/Y(s)$ becomes as follows:

$$\frac{Z(s)}{Y(s)} = \frac{T_1 s}{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1}$$

A–4–10.

Considering small deviations from steady-state operation, draw a block diagram of the air heating system shown in Figure 4–38. Assume that the heat loss to the surroundings and the heat capacitance of the metal parts of the heater are negligible.

Solution. Let us define $\bar{\Theta}_i$ = steady-state temperature of inlet air, °C

$\bar{\Theta}_o$ = steady-state temperature of outlet air, °C M = mass of air in the heating chamber, kg

G = mass flow rate of air through heating chamber, kg/sec c = specific heat of air, kcal/kg °C

R = thermal resistance, °C sec/kcal $\bar{H}$ = steady-state heat input, kcal/sec

C = thermal capacitance of air contained in the heating chamber = Mc, kcal/°C

Let us assume that the heat input is suddenly changed from $\bar{H}$ to $\bar{H} + h$ and the inlet air temperature is suddenly changed from $\bar{\Theta}_i$ to $\bar{\Theta}_i + \theta_i$. Then the outlet air temperature will be changed from $\bar{\Theta}_o$ to $\bar{\Theta}_o + \theta_o$.

The equation describing the system behavior is $C\,d\theta_o = [h + Gc(\theta_i - \theta_o)]\,dt$

or $C\frac{d\theta_o}{dt} = h + Gc(\theta_i - \theta_o)$ Noting that $Gc = \frac{1}{R}$

we obtain $C\frac{d\theta_o}{dt} = h + \frac{1}{R}(\theta_i - \theta_o)$ or $RC\frac{d\theta_o}{dt} + \theta_o = Rh + \theta_i$

Taking the Laplace transforms of both sides of this last equation and substituting the initial condition that $\theta_o(0) = 0$, we obtain $\Theta_o(s) = \frac{R}{RCs + 1}H(s) + \frac{1}{RCs + 1}\Theta_i(s)$

The block diagram of the system corresponding to this equation is shown in Figure 4–39.

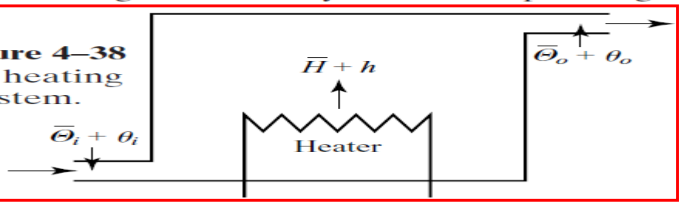


Figure 4–38 Air heating system.

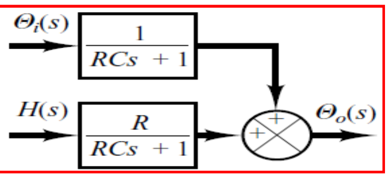


Figure 4–39 Block diagram of the air heating system shown in Figure 4–38.

A–4–11.

Consider the thin, glass-wall, mercury thermometer system shown in Figure 4–40. Assume that the thermometer is at a uniform temperature $\bar{\Theta}$ (ambient temperature) and that at $t = 0$ it is immersed in a bath of temperature $\bar{\Theta} + \theta_b$, where θ_b is the bath temperature (which may be constant or changing) measured from the ambient temperature $\bar{\Theta}$. Define the instantaneous thermometer temperature by $\bar{\Theta} + \theta$, so that θ is the change in the thermometer temperature satisfying the condition that $\theta(0) = 0$. Obtain a mathematical model for the system. Also obtain an electrical analog of the thermometer system.

Solution. A mathematical model for the system can be derived by considering heat balance as follows: The heat entering the thermometer during dt sec is $q\,dt$, where q is the heat flow rate to the thermometer. This heat is stored in the thermal capacitance C of the thermometer, thereby raising its temperature by $d\theta$. Thus the heat-balance equation is $C\,d\theta = q\,dt$ (4–48)

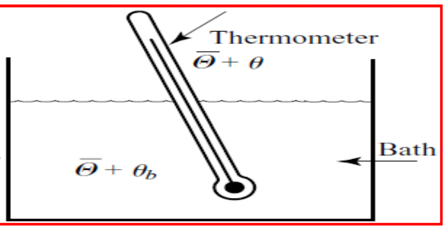


Figure 4–40 Thin, glass-wall, mercury thermometer system.

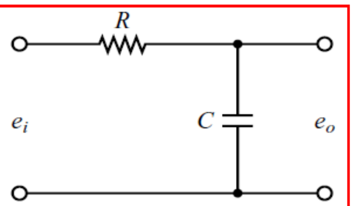


Figure 4–41 Electrical analog of the thermometer system shown in Figure 4–40.

Since thermal resistance R may be written as $R = \frac{d(\Delta\theta)}{dq} = \frac{\Delta\theta}{q}$

heat flow rate q may be given, in terms of thermal resistance R, as $q = \frac{(\bar{\Theta} + \theta_b) - (\bar{\Theta} + \theta)}{R} = \frac{\theta_b - \theta}{R}$

where $\bar{\Theta} + \theta_b$ is the bath temperature and $\bar{\Theta} + \theta$ is the thermometer temperature. Hence, we can rewrite Equation (4–48) as $C\frac{d\theta}{dt} = \frac{\theta_b - \theta}{R}$ or $RC\frac{d\theta}{dt} + \theta = \theta_b$ (4–49)

Equation (4–49) is a mathematical model of the thermometer system.

Referring to Equation (4–49), an electrical analog for the thermometer system can be written as

$$RC\frac{de_o}{dt} + e_o = e_i$$

An electrical circuit represented by this last equation is shown in Figure 4–41.

PROBLEMS

B–4–1. Consider the conical water-tank system shown in Figure 4–42. The flow through the valve is turbulent and is related to the head H by

$$Q = 0.005\sqrt{H}$$

where Q is the flow rate measured in m^3/sec and H is in meters.

Suppose that the head is 2 m at $t = 0$. What will be the head at $t = 60$ sec?

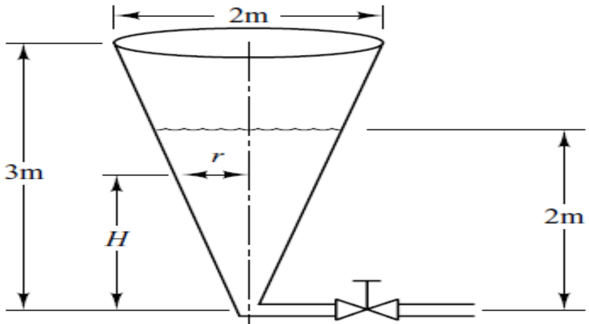


Figure 4–42 Conical water-tank system.

B–4–2. Consider the liquid-level control system shown in Figure 4–43. The controller is of the proportional type. The set point of the controller is fixed.

Draw a block diagram of the system, assuming that changes in the variables are small. Obtain the transfer function between the level of the second tank and the disturbance input q_d. Obtain the steady-state error when the disturbance q_d is a unit-step function.

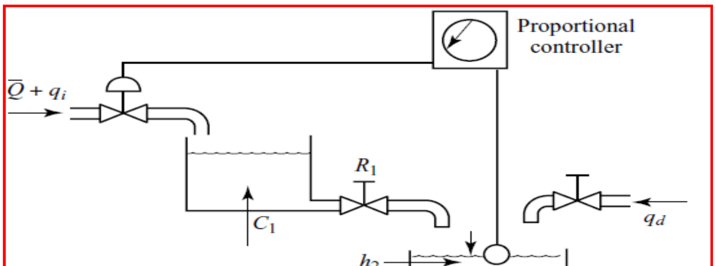


Figure 4–43 Liquid-level control system.

B–4–3. For the pneumatic system shown in Figure 4–44, assume that steady-state values of the air pressure and the displacement of the bellows are $\bar{P}$ and $\bar{X}$, respectively. Assume also that the input pressure is changed from $\bar{P}$ to $\bar{P} + p_i$, where p_i is a small change in the input pressure. This change will cause the displacement of the bellows to change a small amount x. Assuming that the capacitance of the bellows is C and the resistance of the valve is R, obtain the transfer function relating x and p_i.

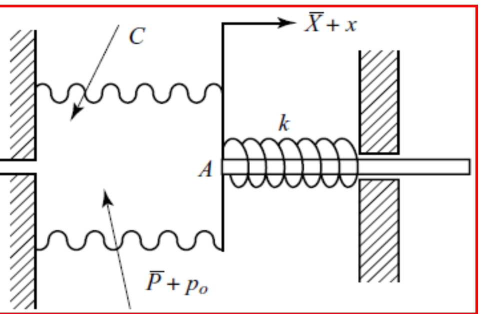


Figure 4–44 Pneumatic system.

B–4–4. Figure 4–45 shows a pneumatic controller. The pneumatic relay has the characteristic that $p_c = Kp_b$, where $K > 0$. What kind of control action does this controller produce? Derive the transfer function $P_c(s)/E(s)$.

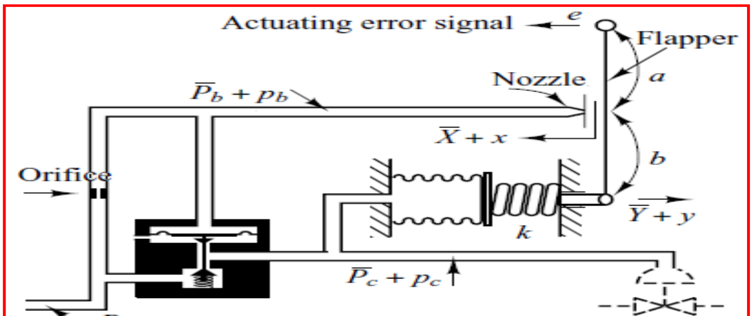


Figure 4–45 Pneumatic controller.

B–4–5. Consider the pneumatic controller shown in Figure 4–46. Assuming that the pneumatic relay has the characteristics that $p_c = Kp_b$ (where $K > 0$), determine the control action of this controller. The input to the controller is e and the output is p_c.

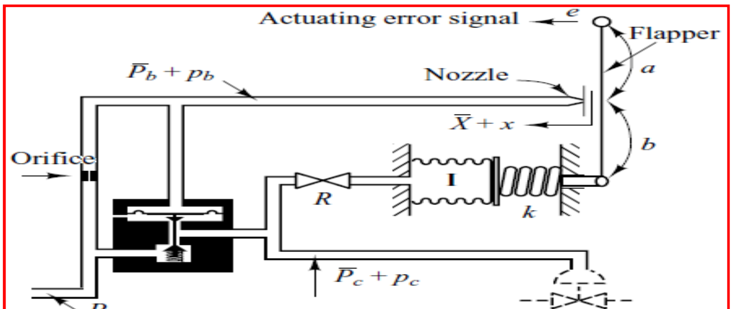


Figure 4–46 Pneumatic controller.

B–4–6. Figure 4–47 shows a pneumatic controller. The signal e is the input and the change in the control pressure p_c is the output. Obtain the transfer function $P_c(s)/E(s)$. Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

B–4–7. Consider the pneumatic controller shown in Figure 4–48. What control action does this controller produce? Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

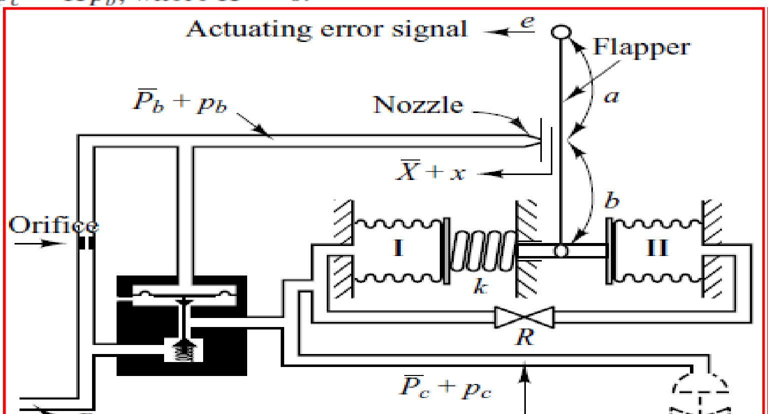


Figure 4–47 Pneumatic controller.

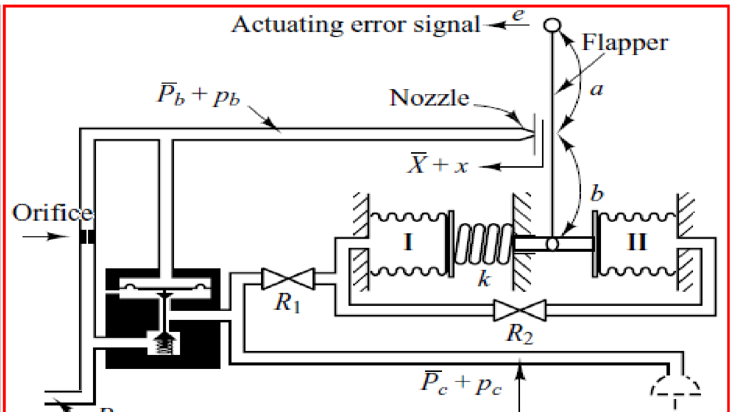


Figure 4–48 Pneumatic controller.

B–4–8. Figure 4–49 shows a flapper valve. It is placed between two opposing nozzles. If the flapper is moved slightly to the right, the pressure unbalance occurs in the nozzles and the power piston moves to the left, and vice versa. Such a device is frequently used in hydraulic servos as the first-stage valve in two-stage servovalves. This usage occurs because considerable force may be needed to stroke larger spool valves that result from the steady-state flow force. To reduce or compensate this force, two-stage valve configuration is often employed; a flapper valve or jet pipe is used as the first-stage valve to provide a necessary force to stroke the second-stage spool valve.

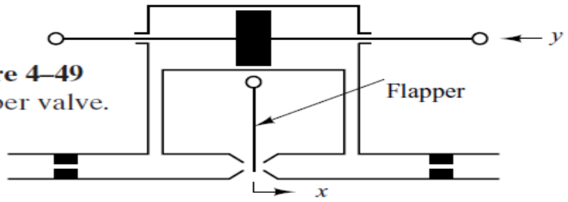


Figure 4–49 Flapper valve.

Figure 4–50 shows a schematic diagram of a hydraulic servomotor in which the error signal is amplified in two stages using a jet pipe and a pilot valve. Draw a block diagram of the system of Figure 4–50 and then find the transfer function between y and x, where x is the air pressure and y is the displacement of the power piston.

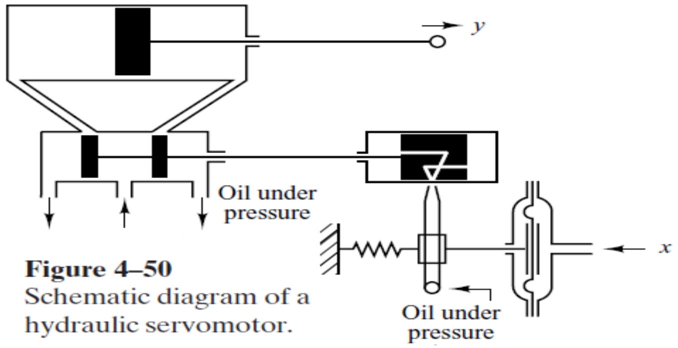


Figure 4–50 Schematic diagram of a hydraulic servomotor.

B–4–9. Figure 4–51 is a schematic diagram of an aircraft elevator control system. The input to the system is the deflection angle θ of the control lever, and the output is the elevator angle ϕ. Assume that angles θ and ϕ are relatively small. Show that for each angle θ of the control lever there is a corresponding (steady-state) elevator angle ϕ.

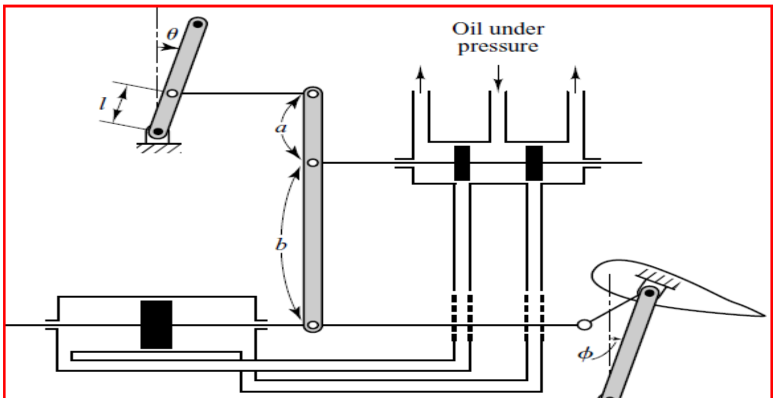


Figure 4–51 Aircraft elevator control system.

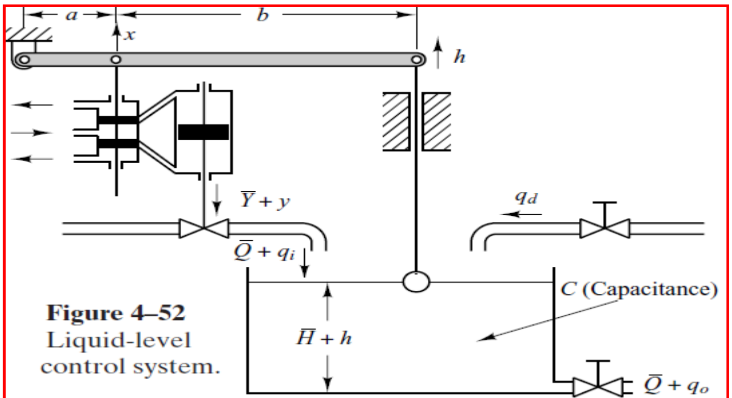


Figure 4–52 Liquid-level control system.

B–4–10. Consider the liquid-level control system shown in Figure 4–52. The inlet valve is controlled by a hydraulic integral controller. Assume that the steady-state inflow rate is $\bar{Q}$ and steady-state outflow rate is also $\bar{Q}$, the steady-state head is $\bar{H}$, steady-state pilot valve displacement is $\bar{X} = 0$, and steady-state valve position is $\bar{Y}$. We assume that the set point $\bar{R}$ corresponds to the steady-state head $\bar{H}$. The set point is fixed. Assume also that the disturbance inflow rate q_d, which is a small quantity, is applied to the water tank at $t = 0$. This disturbance causes the head to change from $\bar{H}$ to $\bar{H} + h$. This change results in a change in the outflow rate by q_o. Through the hydraulic controller, the change in head causes a change in the inflow rate from $\bar{Q}$ to $\bar{Q} + q_i$. (The integral controller tends to keep the head constant as much as possible in the presence of disturbances.) We assume that all changes are of small quantities.

We assume that the velocity of the power piston (valve) is proportional to pilot-valve displacement x, or

$$\frac{dy}{dt} = K_1 x$$

where K_1 is a positive constant. We also assume that the change in the inflow rate q_i is negatively proportional to the change in the valve opening y, or $q_i = -K_v y$

where K_v is a positive constant.

Assuming the following numerical values for the system,

$C = 2\ \text{m}^2$, $R = 0.5\ \text{sec/m}^2$, $K_v = 1\ \text{m}^2/\text{sec}$

$a = 0.25\ \text{m}$, $b = 0.75\ \text{m}$, $K_1 = 4\ \text{sec}^{-1}$

obtain the transfer function $H(s)/Q_d(s)$.

B–4–11. Consider the controller shown in Figure 4–53. The input is the air pressure p_i measured from some steady-state reference pressure $\bar{P}$ and the output is the displacement y of the power piston. Obtain the transfer function $Y(s)/P_i(s)$.

B–4–12. A thermocouple has a time constant of 2 sec. A thermal well has a time constant of 30 sec. When the thermocouple is inserted into the well, this temperature-measuring device can be considered a two-capacitance system.

Determine the time constants of the combined thermocouple–thermal-well system. Assume that the weight of the thermocouple is 8 g and the weight of the thermal well is 40 g. Assume also that the specific heats of the thermocouple and thermal well are the same.

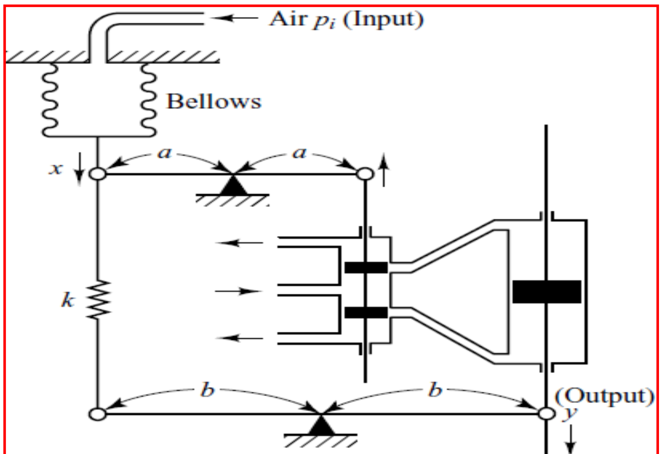


Figure 4–53 Controller.

************* **End of Chapter (4)** *************